HOMOTOPICALLY TRIVIALIZING THE CIRCLE IN THE FRAMED LITTLE DISKS

GABRIEL C. DRUMMOND-COLE

Abstract. This paper confirms the following suggestion of Kontsevich. In the appropriate derived sense, an action of the framed little disks operad and a trivialization of the circle action is the same information as an action of the Deligne-Mumford-Knudsen operad. This improves an earlier result of the author and Bruno Vallette.

1. Introduction

Configurations of points and disks on a surface have a long history of study in a number of different branches of mathematics. In some situations, spaces of such configurations can parameterize operations in topological spaces, modules over a ring, or more generally objects in a symmetric monoidal category. In this point of view, the number of points or disks keeps track of the number of inputs and outputs of the parameterized operation. Kontsevich suggested [Kon05] that a specific relationship should hold between two such operation spaces.

The first set of operation spaces is the framed little disks [Get94a], the noncompact moduli space of genus zero surfaces with parameterized (marked) boundary. The framed little disks generate a subcategory of a certain two-dimensional cobordism category, where the connected morphisms are genus zero surfaces from some number of "incoming" circle boundary components to a single "outgoing" circle, and so these operations arise as the genus zero, n-to-one operations of a two dimensional topological conformal field theory. There are evident circle actions in this setting by rotating the boundary circles.

The second set of operation spaces is the Deligne-Mumford-Knudsen spaces [Knu83], which constitute a compactification of the moduli space of genus zero surfaces with marked points. An action of these spaces can be considered as the genus zero, n-to-one operations of a different kind of field theory.

The relationship between these two spaces of operations, at this genus zero, n-to-one level, is that an action of the framed little disks can be extended to an action of the Deligne-Mumford-Knudsen spaces if the action of the circle is homotopically trivial, and that the data of such a trivialization of the circle gives rise to an action of the Deligne-Mumford-Knudsen spaces (and homotopically no additional information). This paper works through this statement rigorously, using the language of operads. Such homotopical trivializations of the circle arise in practice. For example, the action of the framed little disks in the field theory determined by the

Date: December 7, 2011.
This material is based upon work supported by the National Science Foundation under Award No. DMS-1004625.

category of sheaves on a smooth projective Calabi-Yau manifold has a homotopically trivial circle action, using Keller [Kel98].

The main theorem is the following:

Theorem 1.1. *The Deligne-Mumford-Knudsen genus zero operad $\overline{\mathcal{M}}$ is a realization of the homotopy pushout of the following diagram of operads.*

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & I \\ FLD & \end{tikzcd}$$

Here S^1 is the circle, I is the trivial operad, and FLD is the operad of framed little disks.

This implies the following result at the level of representations:

Corollary 1.2. *For any sufficiently cofibrant models $\widetilde{FLD}$ and $\widetilde{\overline{\mathcal{M}}}$ of the framed little disks and Deligne-Mumford-Knudsen operad, the space of $\widetilde{FLD}$-structures on a space X with trivial $\widetilde{FLD}(1)$-action is weakly equivalent to the space of $\widetilde{\overline{\mathcal{M}}}$-structures on X.*

The main theorem is pleasant, but not unexpected. Different versions of this statement have been discussed by Costello as well as Kontsevich, and a rough outline of a partial argument is present in [Mar], although there is no indication of the homotopy invariance there. A cognate of this theorem was proven at the level of rational homology in [DK] and [DCV], relying on earlier work of Getzler [Get94b, Get95]. This theorem is a nice improvement; using adjunctions along the lines of those in [SS03], one can see that the topological story implies the same result at the level of integral, not just rational, homology.

In order to prove the main theorem, we first describe an operad $\overline{\mathbb{M}}$ which is the ordinary pushout of (one model of) the diagram above, and show that there is a weak equivalence of operads $\overline{\mathcal{M}} \to \overline{\mathbb{M}}$. Then we use another model of the diagram with better cofibrancy properties to generate an operad $\overline{\mathbb{M}}_{\bigcirc}$ which realizes the homotopy pushout of the diagram. This comes with a natural map $\overline{\mathbb{M}}_{\bigcirc} \to \overline{\mathbb{M}}$. The most involved step is showing that this natural map is a weak equivalence.

The paper is organized as follows. Section 2 describes the different versions of the operads that will be involved in the various constructions, and then sections 3 through 5 describe $\overline{\mathbb{M}}$, prove it to be the pushout, and relate it to the Deligne-Mumford-Knudsen operad. Section 6 sets the abstract groundwork necessary to describe $\overline{\mathbb{M}}_{\bigcirc}$ and Section 7 shows that the two pushouts are weakly equivalent. There are several appendices, containing model category theory background, a brief description of the slightly nonstandard conventions used for trees and operads, and a deferred proof from the last section.

Some words on possible extensions are in order. It would be nice to extend this result to the full cobordism category, and it seems likely that a similar statement should be true in that context. The results of Section 6 should hold with little modification, but there seem to be some technical difficulties in generating the correct version of $\overline{\mathbb{M}}$ in that setting. Further, the argument for the equivalence of the two pushouts would almost certainly need to be completely replaced.

It is also natural to wonder how much of this holds for the framed n-balls, for $n > 2$, viewed as a relatively much smaller subcategory of the n-cobordism category. There the results of Section 6 fail completely, as the actions of $SO(n)$ on the framed n-balls are not free, and it seems unlikely that the homotopy quotient and naive quotient should coincide.

The presence of a unit in the framed little disks operad also creates problems for Section 6. The unit disrupts the filtration of the framed little disks by arity, so that the space of one little disk cannot be restricted to the circle alone. This may only be a technical problem, but the unit also has the same problem as higher dimensional balls: the action of the circle is not free. This probably implies a space of units in the homotopy pushout equivalent to BS^1. It is not clear how this should be reflected in (a modification of) the geometric compactification.

Acknowledgements. The author would like to thank Kevin Costello, Joseph Hirsh, and John Terilla for many helpful conversations.

2. Variations on the framed little disks

The purpose of this section is to define the various operads that will be used in the sequel.

Definition 2.1. The group $\mathrm{Aff}\,\mathbb{C}$ is $\mathbb{C} \ltimes GL_1\mathbb{C} \cong \mathbb{C} \ltimes \mathbb{C}^*$, which acts on the complex plane $\mathbb{C}$ by translation, dilation, and rotation.

In coordinates, if the action is taken to be rotation and dilation first, followed by translation, then then the group action is

$$(c_1, r_1) \odot (c_2, r_2) = (c_1 + r_1 c_2, r_1 r_2).$$

The inverse of (c, r) is $(-\frac{c}{r}, \frac{1}{r})$ and the identity is $(0, 1)$.

$\mathrm{Aff}\,\mathbb{C}$ can also be identified with the configuration space of a disk with a marked point on its boundary in the plane; The element (c, r) in this presentation corresponds to the disk centered at c with a marked point on its boundary at $c + r$. In this context, the composition map involves scaling a disk up or down and rotating it. The identity corresponds to the standard disk.

Definition 2.2. Let S be a finite set. The *configuration space of S framed disks in the plane* is:

$$\mathfrak{C}(S) = \{(c_s, r_s)_{s \in S} \in (\mathbb{C} \times \mathbb{C}^*)^S : |c_s - c_{s'}| > |r_s| + |r_{s'}|\}.$$

The condition ensures that every two disks in the plane are disjoint.

The *configuration space of S framed disks or points in the plane* is:

$$\mathfrak{C}^0(S) = \{(c_s, r_s)_{s \in S} \in (\mathbb{C} \times \mathbb{C})^S : |c_s - c_{s'}| > |r_s| + |r_{s'}|\}.$$

Let $\odot$ denote the composition map $(c_1, c_1) \odot (c_2, r_2) \mapsto (c_1 + r_1 c_2, r_1, r_2)$ of $\mathrm{Aff}\,\mathbb{C}$; by abuse of notation, we will use $\odot$ to denote as well its restriction to various related spaces as well.

Definition 2.3. The operad FLD of *framed little disks* consists of the following spaces: $FLD(S)$ is the subset of $\mathfrak{C}(S)$ such that:

1. $|c_s| + |r_s| \leq 1$, and
2. this inequality is strict unless $c_s = 0$.

We will use the custom that $FLD(\emptyset)$ is empty. Note that for $|S|>1$, the conditions imply that $|c_s|+|r_s|<1$.

Geometrically, the c_s correspond to the centers of disjoint little disks in the large disk of radius one in the plane, each having radius $|r_s|$ and with a marked point at c_s+r_s.

The $\mathbb{S}_S$ action is on the factors of $\mathfrak{C}(S)$, and the composition map $\circ_{f,\hat{s}}$ is derived from $\odot$; explicitly, using the shorthand δ_s to refer to the pair (c_s, r_s):

$$\{\delta_s\}_{s\in S}\circ_{f,\hat{s}}\{\delta_{s'}\}_{s'\in S'}=\{\delta_s\}_{s\in S, s\neq\hat{s}}\sqcup\{\delta_{\hat{s}}\odot\delta_{s'}\}_{s'\in S'}$$

using f for relabelling. So we compose $((c_{s'}, r_{s'}))$ into the factor (c,r) by replacing (c,r) with the factors $(c,r)\odot(c_{s'}, r_{s'})$.

We denote by FLD_1 the arity one part of FLD. That is, $FLD(S)$ is $FLD(S)$ if $|S|=1$ and is empty otherwise.

Definition 2.4. The operad $FLD_{\bigcirc}$ of *annulus-free framed little disks* consists of the suboperad of FLD so that if $|S|=1$ then $FLD_{\bigcirc}(S)$ consists only of points on the unit circle, $(0,r)$ with $|r|=1$. This is stable under composition.

Definition 2.5. Let S be an ordered finite set. For convenience, suppose $S=\{1,2,\ldots\}$. The subspaces $FLD^{\mathrm{norm}}_{\bigcirc}(S)$ of $FLD_{\bigcirc}(S)$ consist of those points of $FLD_{\bigcirc}(S)$ where $c_2-c_1\in\mathbb{R}_+$ and $r_s\in\mathbb{R}_+$ (the first condition is empty for $|S|<2$).

Definition 2.6. The operad S^1 is the complex units under multiplication, concentrated in arity one. It can be viewed as a suboperad of $FLD_1\subset FLD$ or $FLD_{\bigcirc}$.

Lemma 2.7. *The composition $FLD^{\mathrm{norm}}_{\bigcirc}(S)\to FLD_{\bigcirc}(S)\to S^1\backslash FLD_{\bigcirc}(S)/(S^1)^S$ is a homeomorphism.*

Proof. Acting on $FLD_{\bigcirc}$ on the left by some element of S^1 and simultaneously on the right by some element $(S^1)^S$ takes c_2-c_1 and r_s to $\mathbb{R}_+$. These elements of S^1 are $\frac{|c_2-c_1|}{c_2-c_1}$ and $\frac{|r_s|(c_2-c_1)}{r_s|c_2-c_1|}$. Thus, there is a map from $FLD_{\bigcirc}(S)$ to $S^1\times(S^1)^S$ picking out these elements, which gives a map from $FLD_{\bigcirc}(S)$ to $FLD^{\mathrm{norm}}_{\bigcirc}(S)$ by composing on the left and right by these elements of S^1. This map is stable on the equivalence classes defining the quotient, so descends to an inverse of the map described above. □

Definition 2.8. The *trivialized annuli operad* An^{triv} is the subset of $\mathfrak{C}^0(S)$ such that $|c_s|+|r_s|\leq 1$, concentrated in arity 1. This operad is a "trivialization" of FLD_1 and/or S^1.

Definition 2.9. The collection $\mathcal{M}$ is defined as follows:

$$\mathcal{M}(S)=\begin{cases}\emptyset & |S|<2\\ \mathrm{Emb}(S,\mathbb{C})/\operatorname{Aff}\mathbb{C} & |S|\geq 2\end{cases}.$$

Here $\mathrm{Emb}(S,\mathbb{C})$ denotes embeddings of S, viewed as a discrete space, into $\mathbb{C}$.

Definition 2.10. The collection of *Deligne-Mumford-Knudsen sets* $\overline{\mathcal{M}}$ is defined as follows:

$$\overline{\mathcal{M}}(S)=\coprod_{T\in\mathbb{T}(S)}\prod_{v\in V(T)}\mathcal{M}(\mathrm{in}(v))$$

That is, an element of $\overline{\mathcal{M}}(S)$ is a tree whose vertices are each decorated with an embedding of $\mathrm{in}(v)$ into $\mathbb{C}$ up to the action of the affine group of $\mathbb{C}$.

Definition 2.11. The *realization* of $\prod \mathbf{x}_v$ in $\overline{\mathcal{M}}(S)$ is a topological space with extra structure obtained as follows. For each vertex, take a copy of complex projective space, marked at ∞ and by the points of $\mathbf{x}_v$ up to $\operatorname{Aff}\mathbb{C}$. If there is an edge between two vertices, identify ∞ on one copy of complex projective space with the correct point of the other. Remove the points S corresponding to the leaves from this quotient, and give it the subspace topology. The topological structure does not depend on any choices, and because everything is taken up to conformal automorphisms, this space has a conformal structure away from the identified points (which are called *nodal points*).

Definition 2.12. A *contraction* from one point to another in $\overline{\mathcal{M}}(S)$ is a map of realizations which respects S, is a homeomorphism except possibly at nodal points of the realization of the range, and so that the preimage of a nodal point is either a nodal point or a circle containing no nodal points.

Definition 2.13. $\overline{\mathcal{M}}(S)$ is a topological space with a topology with a local basis as follows. Let $\mathbf{x} = \prod \mathbf{x}_v$ be an element of $\overline{\mathcal{M}}(S)$. $\mathbf{x}_v$ is a configuration of points up to $\operatorname{Aff}\mathbb{C}$. Let U_v be a bounded open subset of $\mathbb{C}$ (taken up to $\operatorname{Aff}\mathbb{C}$) whose closure does not contain any $\mathbf{x}_v$. The neighborhood $\mathscr{N}_{\mathbf{x},U}$ consists of those points y in $\overline{\mathcal{M}}(S)$ which have a contraction to $\mathbf{x}$ which is conformal on the preimage of U_v.

This presentation of the topology of $\overline{\mathcal{M}}(S)$ is derived from [HV10].

Fact 1. *$\overline{\mathcal{M}}$ has the structure of an operad, where the unit is the tree with one leaf and no vertices and partial composition is given by grafting trees.*

3. Characterizing a pushout

The purpose of the next two sections is to describe the pushout of the diagram $FLD \leftarrow FLD_1 \rightarrow An^{\text{triv}}$ explicitly. We will describe it as an operad of decorated trees, and to that point will define a marking for vertices of a tree.

Definition 3.1. Let v be a vertex of a tree T. If v is the root, then the marking set Mark_v is the subset of $\mathfrak{C}(E(v))$ such that:

(1) $0 < |r_s|$ if r_s is in a factor (c_s, r_s) indexed by an external edge of T.
(2) $r_s = 0$ if r_s is in a factor indexed by an internal edge of T,
(3) $|c_s| + |r_s| \leq 1$ for every factor in the product, and
(4) this inequality is strict unless $c_s = 0$.

If v is not the root, then Mark_v is the subset of $\mathfrak{C}(E(v))/\operatorname{Aff}\mathbb{C}$ such that

(1) $0 < |r_s|$ if r_s is in a factor indexed by an external edge of T, and
(2) $r_s = 0$ if r_s is in a factor indexed by an internal edge of T.

Here, the $\operatorname{Aff}\mathbb{C}$ action is on all factors of $\mathfrak{C}(E(v))$ by left multiplication with $\odot$.

Both marking sets are configuration spaces of disjoint little disks indexed by the external edges in centered at c_s of radius $|r_s|$ with one marked point on the boundary circle at $c_s + r_s$ along with additional disjoint marked points at c_s indexed by the internal edges.

The marking for the root is such configurations in the disk; the marking for other vertices is such configurations in the plane up to conformal automorphisms of the plane.

Definition 3.2. The trivialized little disks operad $\overline{\mathbb{M}}$ consists of the sets (for now) $\overline{\mathbb{M}}(S)$ where $\overline{\mathbb{M}}(S)$ is

$$\coprod_T \prod_{v \in V(T)} \mathrm{Mark}_v.$$

Here the disjoint union is over nontrivial nearly stable S-trees T.

Proposition 3.3. *$\overline{\mathbb{M}}$ carries a well-defined operad structure in the category of sets. As an operad of sets, it is the pushout of the trivialized annuli operad An^{triv} and the framed little disks FLD over $FLD_1 \subset An^{\mathrm{triv}}$:*

$$\begin{tikzcd}
FLD_1 \arrow[r] \arrow[d] & An^{\mathrm{triv}} \arrow[d] \arrow[ddr] & \\
FLD \arrow[r] \arrow[drr] & \overline{\mathbb{M}} \arrow[dr, dotted] & \\
& & \mathcal{O}
\end{tikzcd}$$

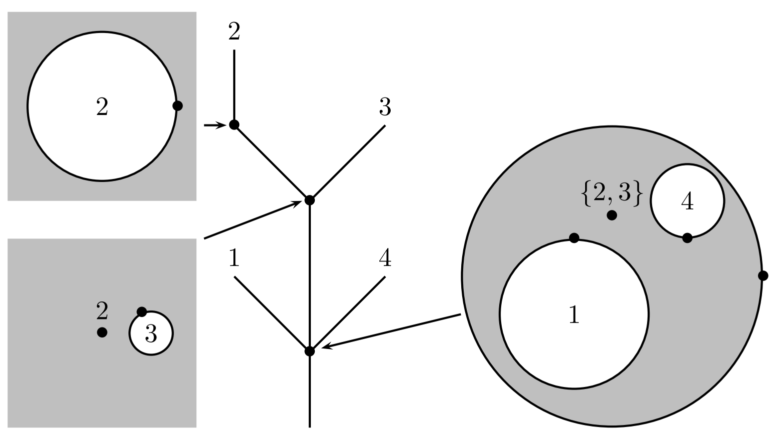

FIGURE 1. An element of $\overline{\mathbb{M}}(\{1, 2, 3, 4\})$

Proof of Proposition 3.3. The composition in $\overline{\mathbb{M}}$ is as follows: to glue a disk into a little disk in the plane up to conformal automorphism, scale the disk and glue it in using $\odot$. This is independent of the conformal representative. The other vertices of the tree involved do not change any decorations. If the resulting configuration in the plane is unstable (a single point) then forget the corresponding tree vertex.

Given this composition structure, the tree with one leaf and one vertex decorated by $(0, 1)$ is a unit for composition. It is routine to verify that composition is associative, since $\odot$ is.

The configuration space in $\overline{\mathbb{M}}$ for a S-corolla is in canonical bijection with $FLD(S)$. Composition of corollas in $\overline{\mathbb{M}}$ agrees with composition in the framed little disks. This discussion includes annuli of positive radius.

The annulus $(c, 0)$ maps to the 1-tree with two vertices, the root decorated by $(c, 0)$ and the other vertex decorated by a single little disk (c, r) up to $\mathrm{Aff}\,\mathbb{C}$ (which is no information, as this is the quotient of $\mathrm{Aff}\,\mathbb{C}$ by itself). A straightforward

check demonstrates that composition of two radius zero annuli, or of a radius zero annulus with a positive radius annulus agrees with composition in $\overline{\mathbb{M}}$ on both the right and the left.

This establishes the operad structure and how $\overline{\mathbb{M}}$ fits into the diagram, but not the fact that $\overline{\mathbb{M}}$ is the pushout, which will be shown after establishing a lemma. □

Definition 3.4. Let $\mathbf{x} = ((c_s, r_s))_{s \in S}$ be a point in $FLD(S)$. The *annular image of* $\mathbf{x}$ is the set of little disks of the form

$$(c_0, r_0) \circ \mathbf{x} \circ ((0, r'_s))_{s \in S}$$

for (c_0, r_0) and $(0, r'_s)$ in FLD_1. This composes $\mathbf{x}$ with annuli centered at 0 on the right and an arbitrary annulus on the left (in this case, the result on the left is independent of parenthesization). If S' is a subset of $\{0\} \sqcup S$, the *annular image corresponding to* S' is the subset of the annular image so that $(c_s, r_s) = (0, 1)$ for indices not in the subset. That is, it is the composition with annuli only at the specified leaves and possibly the root.

Lemma 3.5. *The image of FLD and the trivial annulus $(0, 0) \in An^{\mathrm{triv}}$ generate $\overline{\mathbb{M}}$ under operadic composition. In particular, every element of $\overline{\mathbb{M}}(S)$ can be written as a composition of these generators along a tree T with the following characteristics:*

1. *The root and any vertex which is the successor of a leaf is labeled with a framed little disk,*
2. *for each edge in T, one of the input and output is labeled with a framed little disk and the other is labeled with $(0, 0)$, and*
3. *if all vertices labeled with $(0, 0)$ are forgotten, then the resultant tree is nearly stable.*

Any two such decompositions have the same underlying tree. If there are two decompositions with vertex labels $\mathbf{x}_v$ and $\mathbf{x}'_v$ for $v \in V(T)$, then $\mathbf{x}_v = (0, 0)$ if and only if $\mathbf{x}'_v = (0, 0)$; if $\mathbf{x}_v$ is a label from the framed little disks, then $\mathbf{x}'_v$ is as well and $\mathbf{x}_v$ and $\mathbf{x}'_v$ share a configuration in their annular images corresponding to the internal edges of the vertex v.

On the other hand, if $\mathbf{x}'_v$ is in the annular image of $\mathbf{x}_v$ corresponding to the internal edges of v for each vertex v decorated with a framed little disk configuration, then the composition of the $\mathbf{x}_v$ along T and $\mathbf{x}'_v$ along T is the same element of $\overline{\mathbb{M}}$.

Proof. Consider an element $\mathbf{x}$ of $\overline{\mathbb{M}}$ with underlying tree T. That is, $\mathbf{x}$ is a product over $v \in V(T)$ of $\mathbf{x}_v \in \mathrm{Mark}_v$. Let T' be the tree obtained by inserting a vertex on every internal edge of T; we will describe $\mathbf{x}$ as a composition of the generators along T'. To do this, for each vertex in T', we must supply a label from our generating set. Label each newly inserted vertex with $(0, 0)$.

For the root vertex, $\mathbf{x}_v$ is a configuration of disks and points in the plane. Arbitrarily choose small radii for each point; this is possible because the conditions defining Mark_v are open. This gives an element of FLD to label the root vertex of T'.

For the other vertices of T, fix a representative under the $\mathrm{Aff}\,\mathbb{C}$ action wholly contained in the interior of the unit disk and perform the same procedure.

Composing along T' with the given labels yields $\mathbf{x}$. It is clear that T' is determined by T. This argument also shows that if $\mathbf{x}'_v$ is in the annular image of $\mathbf{x}_v$ corresponding to the appropriate edges, then $\mathbf{x}'_v$ and $\mathbf{x}_v$ differ only by a choice of $\mathrm{Aff}\,\mathbb{C}$ representative and nonzero radii.

The only thing remaining is to show uniqueness of the labels from the framed little disks. For two decorations on the root of T' to yield the same decoration on the root of T, the only thing that can differ between them are the radii corresponding to internal edges. The decoration where the radii are the minimum of the two is in the annular image of both.

For two decorations on another vertex of T' to yield the same decoration on the corresponding vertex of T, they may differ, as before, in the radii corresponding to internal edges, but may also differ by the overall action of an element of $\mathrm{Aff}\,\mathbb{C}$ acting on the left. However, it is easy to see that $FLD_1^{-1} \odot FLD_1$ yields all of $\mathrm{Aff}\,\mathbb{C}$. Then using any factorization of the element of $\mathrm{Aff}\,\mathbb{C}$ acting on the left along with the argument used for the root show that the two decorations share a point in their annular image. $\square$

End of proof of Proposition 3.3. Consider a decomposition as in Lemma 3.5. Suppose that at some vertex other than the root, we make two different choices of framed little disk, $\mathbf{x}_v$ and $\delta \odot \mathbf{x}_v$, where δ is an annulus. Because this is not the root, it will be composed in the composition with $(0, 0)$ on the left. If by abuse of notation we denote all the given maps into $\mathcal{O}$ by Φ, then we are comparing $\Phi((0, 0)) \circ \Phi(\mathbf{x}_v)$ with $\Phi((0, 0)) \circ \Phi(\delta \odot \mathbf{x}_v)$ Because $\Phi : FLD \to \mathcal{O}$ is a map of operads, the latter is equal to $\Phi((0, 0)) \circ \Phi(\delta) \circ \Phi(\mathbf{x}_v)$; because the maps commute over annuli, this is the same as $\Phi((0, 0)) \circ \delta \circ \Phi(\mathbf{x}_v) = \Phi((0, 0)) \circ \Phi(\mathbf{x}_v)$, as desired. The other type of ambiguity is similar.

This shows that the decomposition of Lemma 3.5 gives a uniquely determined map of collections from $\overline{\mathbb{M}} \to \mathcal{O}$ given coherent maps of operads $FLD \to \mathcal{O}$ and $An^{\mathrm{triv}} \to \mathcal{O}$. A straightforward check of several cases verifies that the induced map behaves well with respect to composition in $\overline{\mathbb{M}}$. $\square$

4. The pushout in spaces

The following is the main theorem of this section.

Proposition 4.1. *There are topologies on the sets $\overline{\mathbb{M}}(S)$, described below, which make $\overline{\mathbb{M}}$ a topological operad, make the embeddings of the trivialized annuli and the framed little disks continuous, and realize $\overline{\mathbb{M}}$ as the pushout in the category of topological operads.*

Description of the topology of Proposition 4.1. To begin with, we shall contain the marking set Mark_v into a larger ambient space.

Definition 4.2. Let v be a vertex of a tree T. If v is the root, then the *augmented marking space* $\widehat{\mathrm{Mark}}_v$ is the subset of $\mathfrak{C}(E(v))$ such that:

(1) $0 < |r_s|$ if r_s is in a factor (c_s, r_s) indexed by an external edge of T.
(2) $|c_s| + |r_s| \leq 1$ for every factor in the product, and
(3) this inequality is strict unless $c_s = 0$.

If v is not the root, then $\widehat{\mathrm{Mark}}_v$ is the subset of $\mathfrak{C}(E(v))/\,\mathrm{Aff}\,\mathbb{C}$ such that $0 < |r_s|$ if r_s is in a factor indexed by an external edge of T.

The augmented marking space contains the marking set as the subspace with $r_s = 0$ for each factor indexed by an internal edge of T.

Now fix a nontrivial nearly stable S-tree T and fix an ordering of S. We will define a set map

$$\prod_{v \in V(T)} \widehat{\mathrm{Mark}_v} \to \overline{\mathbb{M}}$$

The part of $\overline{\mathbb{M}}(S)$ with underlying tree T looks like $\prod \mathrm{Mark}_v$, and such points will be taken to themselves by this map.

For $\mathbf{x}$ in $\overline{\mathbb{M}}(S)$ with underlying tree T, let U_v be a neighborhood of $\mathbf{x}_v$; then the image of $\prod U_v$ will be an neighborhood of $\mathbf{x}$ set in our basis, called $\mathscr{N}_{\mathbf{x},U}$.

The map works as follows. Let $\mathbf{y} = \prod \mathbf{y}_v$ be a point in $\prod_T \widehat{\mathrm{Mark}_v}$. Let $\mathcal{E}$ be the set of internal edges e of T so that $|r_e| > 0$. Then $\mathbf{y}$ will land in the component of $\overline{\mathbb{M}}(S)$ whose underlying tree is the edge contraction of T along the edges $\mathcal{E}$.

To specify the map, we must provide maps from a product of $\widehat{\mathrm{Mark}_v}$ to a single Mark_v after the contraction. The ordering of S induces one on $\mathrm{in}(v)$; for ease, suppose $\mathrm{in}(v) = \{1, 2, \ldots\}$. If v is not the root[1], then there is a unique representative $\tilde{\mathbf{y}}_v$ in the $\mathrm{Aff}\,\mathbb{C}$ orbit of $\mathbf{y}_v$ with $c_1 = 0$, c_2 on the positive real line, and the Euclidean diameter of the union of the disks of center c_s and radius $|r_s|$ in the plane equal to $\frac{1}{2}$. There is a unique marking with center 0 and radius vector to 1 if v has only one incoming edge (which must then be external and have a nonzero radius). We can view $\tilde{\mathbf{y}}_v$ as a configuration in the disk, rather than the plane. Then the map from a product of $\widehat{\mathrm{Mark}_v}$ to Mark_v is just iterated $\odot$ of the $\tilde{\mathbf{y}}_v$ along the factors of the product specified by the contraction edges. The associativity of $\odot$ ensures that this is independent of the order of contraction.

The point $\mathbf{x}$ has radius vector zero for all internal edges, so is taken to itself, as promised. □

Lemma 4.3. *The set $\{\mathscr{N}_{\mathbf{x},U}\}$ forms the basis for a topology which is independent of the ordering chosen on S.*

Proof. It is clear that these sets cover $\overline{\mathbb{M}}(S)$. $\mathscr{N}_{\mathbf{x},U} \cap \mathscr{N}_{\mathbf{x},U'}$ contains $\mathscr{N}_{\mathbf{x},U\cap U'}$ so it suffices to show that for any $\mathbf{x} \in \mathscr{N}_{\mathbf{z},U}$ there exists a basis element $\mathscr{N}_{\mathbf{x},U}$ centered at $\mathbf{x}$ contained therein. Note that $\mathbf{z}$ lives in a product over the tree T and $\mathbf{x}$ lives in the product over the tree $T_{\mathcal{E}}$, where $\mathcal{E}$ is some set of internal edges of T. For each $U_v \in \widehat{\mathrm{Mark}_v}$, for $v \in V(T)$, consider the open subset ${U_v}'$ specified to be those points of U_v so that factors (c, r) indexed by edges in $\mathcal{E}$ have nonzero radius.

Now we must define open sets $U_v \in \widehat{\mathrm{Mark}_v}$ for each vertex v of the tree $T_{\mathcal{E}}$. Such a vertex is an equivalence classes of vertices in T. Then the open set $U_{\{v_{e_s}\}}$ (where the subscript comprises the elements of such an equivalence class) corresponds to the composition of U'_v by $\odot$ along the factors indexed by contraction edges. As before, we pick a unique normalized representative for the factors in $\widehat{\mathrm{Mark}_v}$ involved in a composition.

It is an exercise to show that the resultant sets U_v are open, as desired.

Changing the ordering on S changes the normalization used in the composition map. It is sufficient to show that if S and $\hat{S}$ are the same set with different ordering, then every neighborhood $\mathscr{N}_{\mathbf{x},U}$ contains a neighborhood $\widehat{\mathscr{N}_{\mathbf{x},U'}}$. The two normalizations differ by a rotation and a translation of less than $\frac{1}{2}$. U contains a

[1]Let $\tilde{\mathbf{y}}_v$ denote $\mathbf{y}_v$ for v the root.

neighborhood of a pair $(c, 0)$ into which the decoration on this vertex is to be composed; this neighborhood must contain the ball consisting of all pairs (c', r') with $|c' - c|$ and $|r'|$ less than ϵ, for some small ϵ. If the corresponding neighborhood in U' is chosen inside the ball of radius, say, $\frac{\epsilon}{2}$, then the composition using the normalization from $\hat{S}$ will be contained in that from S. $\square$

Proof of that the operadic structure maps are continuous. Now that the topology is defined, it should be shown that the various structure maps of the operad are continuous. This is obvious in the case of the unit, and the symmetric group action is continuous because the topology is independent of the ordering chosen to define the topologizing map.

Next, if $\mathbf{x} = \mathbf{y} \circ_{f,s} \mathbf{z}$, then the preimage under $\circ_{f,s}$ of $\mathscr{N}_{\mathbf{x},U}$ should contain a neighborhood of $\mathbf{y} \times \mathbf{z}$. In the case of stable composition of $\mathbf{y}$ and $\mathbf{z}$, this follows directly from the continuity of $\odot$, so we restrict our attention to the unstable case.

In this case, there are special bivalent vertices $\mathfrak{a}$ of $T_{\mathbf{y}}$ and $\mathfrak{b}$ of $T_{\mathbf{z}}$ where the composition occurs. The complement of these two vertices in the disjoint union of these trees' vertex sets is in canonical bijection with the vertex set of $T_{\mathbf{x}}$.

Call vertex immediately below $\mathfrak{a}$ in $T_{\mathbf{y}}$ (and the corresponding vertex in $T_{\mathbf{x}}$) $\mathfrak{c}$. Without loss of generality, we assume that $U_{\mathfrak{c}}$ contains an ϵ-neighborhood of the decoration $\mathbf{x}_{\mathfrak{c}}$ in some fixed normalization that doesn't depend on the decoration of the edge coming from $\mathfrak{a}$.

For the vertices corresponding to $T_{\mathbf{x}}$, $\mathscr{N}_{\mathbf{x},U}$ already includes a choice of open set U_v. At the bivalent vertices, take $U_{\mathfrak{a}} = \widehat{\mathrm{Mark}}_{\mathfrak{a}}$ (which is a single point, the image of $(0, 1)$ under $\mathrm{Aff}\,\mathbb{C}$) and $U_{\mathfrak{b}} = \widehat{\mathrm{Mark}}_{\mathfrak{b}}$.

Define a map

$$\prod_{V_{\mathbf{y}}} \widehat{\mathrm{Mark}}_v \times \prod_{V_{\mathbf{z}}} \widehat{\mathrm{Mark}}_v \to \prod_{V_{\mathbf{x}}} \widehat{\mathrm{Mark}}_v$$

which composes the decoration on $\mathfrak{b}$ into the appropriate spot of the decoration on $\mathfrak{c}$, keeping all other decorations the same. We will show that performing this operation on points in our product of open sets lands in the product definining $\mathscr{N}_{\mathbf{x},U}$ and that the following diagram commutes, completing the proof by demonstrating that $\mathscr{N}_{\mathbf{y},U} \times \mathscr{N}_{\mathbf{z},U}$ is in the preimage of $\mathscr{N}_{\mathbf{x},U}$.

$$\begin{tikzcd}
\prod_{V_{\mathbf{y}}} \widehat{\mathrm{Mark}}_v \times \prod_{V_{\mathbf{z}}} \widehat{\mathrm{Mark}}_v \arrow[r] \arrow[d] & \overline{\mathbb{M}} \times \overline{\mathbb{M}} \arrow[d, "\odot"] \\
\prod_{V_{\mathbf{x}}} \widehat{\mathrm{Mark}}_v \arrow[r] & \overline{\mathbb{M}}
\end{tikzcd}$$

Since by assumption the composition is unstable, the radius of the decoration (c_0, r_0) of $\mathbf{y}_{\mathfrak{c}}$ corresponding to the edge from $\mathfrak{a}$ is 0, and so for any point in the neighborhood $U_{\mathfrak{c}}$, if (c, r) is the cogent decoration we have $|c - c_0| + |r| < \epsilon$. Then for any point $(c_{\mathfrak{b}}, r_{\mathfrak{b}})$ in $U_{\mathfrak{b}} \subset \mathrm{Aff}\,\mathbb{C}$, we have

$$|c + rc_{\mathfrak{b}} - c_0| + |rr_{\mathfrak{b}}| \leq |c - c_0| + |r|(|c_{\mathfrak{b}}| + |r_{\mathfrak{b}}|) \leq |c - c_0| + |r| < \epsilon.$$

This shows that the image of our constructed product neighborhood of $\mathbf{y} \times \mathbf{z}$ is contained in $\prod U_v$. Commutativity of the diagram basically follows from associativity of $\odot$. The marking on $\mathfrak{a}$ is always the identity, and never matters; there are

a couple of easy extra cases where the radius of the salient markings on $\mathfrak{c}$ or $\mathfrak{b}$ is zero. □

Proof that the embeddings are continuous. For the framed little disks, there is nothing to check; the tree describing a point in the image of the framed little disks has no internal edges, so the topology on the image of $FLD(S)$ is just the subspace topology in $(\mathbb{C}\times\mathbb{C})^S$, just as it is in the framed little disks. For the annuli, it must be checked that the preimage of an open set around the image of a radius zero annulus is open in the operad of trivialized annuli. Since such an annulus is described by a tree with one leaf and two vertices, the root and another vertex. The decoration on the leaf edge is unique so the space $\prod \widehat{\mathrm{Mark}_v}$ is homeomorphic to the augmented marking space of the root. A basis for the open sets around the trivial annulus $(c, 0)$ in this space are formed by the subsets

$$\{(c+p,q)\in\mathfrak{C}(\{e\}) : |c+p|+|q|<1, |p|,|q|<\epsilon\}.$$

This is open in $An^{\mathrm{triv}}(\{e\})$ as well. □

Proof of that $\overline{\mathbb{M}}$ is the pushout. Let $\mathcal{O}$ be a topological operad that accepts topological operad maps from An^{triv} and FLD that agree on S^1. Proposition 3.3 indicates that there is a unique morphism of operads of sets from $\overline{\mathbb{M}}\to\mathcal{O}$ which factors both of these maps. To prove the proposition, we must show that the induced set map is continuous.

Let $\mathbf{y}\in\mathcal{O}$ and $\mathbf{x}\in\overline{\mathbb{M}}$ in its preimage under the induced set map. Then $\mathbf{x}$ has an underlying tree T and can be written as a composition along the tree T' obtained by inserting a vertex on every internal edge of T, where the new vertices are labeled by $(0,0)$ and the old vertices are labeled with minor modifications of the decorations of the corresponding vertices of T, as in Lemma 3.5. Let V denote the vertices of T and V' the vertices of T'. The diagram in figure 2 commutes.

Let $\mathscr{N}$ be an open set containing $\mathbf{y}$. The composition along the top and right side is continuous by assumption, so we can come up with open neighborhoods of any preimage of $\mathbf{x}$ in the product of FLD and An^{triv}. Then we can push those maps down the left side of the diagram.

Consider ϵ-neighborhoods around fixed points $\zeta_v\in FLD(\mathrm{in}(v))$ and around $(0,0)$. The image of the product of these neighborhoods under the first two vertical maps on the left contains the ϵ-neighborhood of the image of the appropriate composition of ζ_v and $(0,0)$. Then the composition of these two maps is open, so pushing forward the preimage of $\mathscr{N}$ we get an open neighborhood of $\mathbf{x}$ in $\prod\overline{\mathbb{M}}$. □

5. The relation to the Deligne-Mumford-Knudsen compactification

The purpose of this section is to relate the pushout operad $\overline{\mathbb{M}}$ to the Deligne-Mumford-Knudsen operad $\overline{\mathcal{M}}$ as follows:

Theorem 5.1. *There is a map of operads $\overline{\mathcal{M}}\to\overline{\mathbb{M}}$ which is the inclusion map of a deformation retraction of collections.*

The proof will follow from three propositions related to a suboperad $\overline{\mathbb{M}}_{\mathrm{DMK}}$ of the pushout operad $\overline{\mathbb{M}}$. After defining $\overline{\mathbb{M}}_{\mathrm{DMK}}$, we shall prove:

Proposition 5.2. *For $|S|>1$, $\overline{\mathbb{M}}_{\mathrm{DMK}}(S)$ is locally homeomorphic to $\mathbb{R}^{2|S|-4}$.*

Proposition 5.3. *$\overline{\mathbb{M}}_{\mathrm{DMK}}(S)$ is a deformation retract of $\overline{\mathbb{M}}(S)$.*

$$\begin{tikzcd}
\prod_{v\in V} FLD(\mathrm{in}(v)) \times \prod_{v\in V'\setminus V} An^{\mathrm{triv}}(\mathrm{in}(v)) \arrow[r] \arrow[d] & \prod_{v\in V'} \mathcal{O} \arrow[ddd, "\text{compose along } T'"] \\
\prod_{v\in V'} \overline{\mathbb{M}} \arrow[d, "\text{compose along edges out of vertices in} V'\setminus V"] & \\
\prod_{v\in V} \overline{\mathbb{M}} \arrow[d, "\text{compose along } T"] & \\
\overline{\mathbb{M}} \arrow[r] & \mathcal{O}
\end{tikzcd}$$

Figure 2.

Proposition 5.4. *There is a bijective map of topological operads:*

$$\Psi : \overline{\mathbb{M}}_{\mathrm{DMK}} \to \overline{\mathcal{M}}.$$

For the latter two, the $|S| = 1$ case is obvious, and for most of the rest of the section we shall assume $|S| > 1$ without comment.

Proof of Theorem 5.1. $\overline{\mathcal{M}}(S) = \overline{\mathcal{M}}_{0,|S|+1}$ is a manifold of dimension $2(|S|+1) - 6 = 2|S| - 4$ as well. Invariance of domain says that an injective continuous map from a space modelled locally by $\mathbb{R}^{2|S|-4}$ to a manifold of the same dimension is a homeomorphism onto its image. Therefore Ψ is an isomorphism of topological operads. This combined with Proposition 5.3 gives the deformation retraction. The inclusion map is a composition of operad maps, so it is an operad map. □

Definition 5.5. $\overline{\mathbb{M}}_{\mathrm{DMK}}(S)$ is the subspace of $\overline{\mathbb{M}}(S)$ characterized by the following:

1. If $|S| = 1$, then $\overline{\mathbb{M}}_{\mathrm{DMK}}(S)$ is just the identity configuration $(0, 1)$ in $\overline{\mathbb{M}}(S)$,
2. if $|S| > 1$, then the tree underlying any element of $\overline{\mathbb{M}}_{\mathrm{DMK}}(S)$ must have edges of the form $(\{s\}, 1)$ for $s \in S$ and $(S, 1)$, and
3. the decoration of the root is the configuration $(0, 0)$.

Lemma 5.6. $\overline{\mathbb{M}}_{\mathrm{DMK}}$ *is a suboperad of* $\overline{\mathbb{M}}$.

Proof. Any compostion of two non-identity elements of $\overline{\mathbb{M}}_{\mathrm{DMK}}$ must involve the unstable grafting of a leaf of one to the root of the other. The other root and the other leaves are not involved in this composition and remain bivalent after it. □

Definition 5.7. Let v be a vertex of a tree T. If v is the root, the *Deligne-Mumford-Knudsen marking space* $\overline{\mathrm{Mark}}_v$ is the singleton $\{(0,0)\}$. Otherwise, $\overline{\mathrm{Mark}}_v$ is the subspace of $\mathfrak{C}(E(v))/\operatorname{Aff}\mathbb{C}$ such that $r_e = 0$ for every factor (c_e, r_e) indexed by an edge coming from a bivalent vertex.

The following follows directly from the definition:

Lemma 5.8. *Let* $\mathbf{x}$ *be in* $\overline{\mathbb{M}}_{\mathrm{DMK}}(S)$ *with underlying tree* T *(and a fixed order on* S*). For* v *a vertex of* T*, let* $\mathbf{x}_v$ *be the decoration on* v*. Let* U_v *denote an open set containing* $\mathbf{x}_v$ *in* $\widehat{\mathrm{Mark}}_v$*. Then the images of sets of the form* $\prod U_v$ *under the map*

$$\prod \widehat{\mathrm{Mark}}_v \to \overline{\mathbb{M}}(S)$$

land in $\overline{\mathbb{M}}_{\mathrm{DMK}}$ *and form a basis for the subspace topology on* $\overline{\mathbb{M}}_{\mathrm{DMK}}$.

Lemma 5.9. *For a fixed* $\mathbf{x} \in \overline{\mathbb{M}}_{\mathrm{DMK}}(S)$ *and fixed ordering of* S*, the map from* $\prod \overline{\mathrm{Mark}}_v \to \overline{\mathbb{M}}_{\mathrm{DMK}}$ *is injective.*

Proof. We want to show that a configuration in $\overline{\mathbb{M}}_{\mathrm{DMK}}$ uniquely determines the decorations on the vertices that were identified to make it. Let T be the underlying tree of $\mathbf{x}$.

Let $\mathbf{y}$ be a point in the image of the map determined by $\mathbf{x}$; $\mathbf{y}$ determines an edge contraction set $\mathcal{E}$. Let $S = \{v_s\}$ be a vertex in the contracted tree $T_{\mathcal{E}}$. we would like to recover the decorations on the vertices v_s involved in the contraction from the decoration on the contracted vertex. We will proceed downward through the set of contracted vertices, starting with the topmost vertices of S.

Assume we have recovered the decoration on every vertex above the vertex v; we wish to recover the decoration $\mathbf{x}_v$ of v from the contraction vertex decoration $\mathbf{y}_S$. Essentially, we know that the decoration on S is equal to a composition, so by inverting the composition map, which is just $\odot$, where we can, we will be able to recover the marking on v.

By assumption, we know all the markings above v, so we can compose the appropriate representatives to get a configuration $\mathbf{x}_v^e$ of points in the disk for each incoming internal edge e of v coming from S. There are only points because at the top level, every disk must have radius 0. Consider each $\mathbf{x}_v^e$ with the normalization used for composition. Because c_1 in each representative is 0, the configuration $\mathbf{x}_v^e$ has a marked point at 0. It has another marked point at $c_1 \neq 0$. In order that $(c,r)\odot\mathbf{x}_v^e = (c',r')\odot\mathbf{x}_v^e$, we must have $c+0r = c'+0r$ so $c = c'$ and $c+rc_1 = c'+r'c_1$ so $r = r'$. This shows that we can uniquely factor $\mathbf{y}_S$ as the composition of some $\mathbf{z}$ with the various $\mathbf{x}_v^e$. This gives us a center and radius vector in $\mathbf{z}$ for each incoming edge of v. Now we want to decompose $\mathbf{z}$ as the composition of some $\mathbf{z}'$ and $\mathbf{x}_v$. We know which centers and radii in $\mathbf{z}$ sitting in the standard disk came from $\mathbf{x}_v$, including c_0 and c_1, the image of 0 and a positive real. We can also measure the Euclidean diameter of the union of the disks involved. So setting $(c,r)\odot\mathbf{x}_v$ to be the points and distances we know, we get the equations

$$c + 0r = c_0,\ c_1 \in c + r\mathbb{R}_+,$$

and the diameter is $\frac{|r|}{2}$. These uniquely specify c and $r \neq 0$ and we can compose on the left by the inverse to (c,r) to obtain $\mathbf{x}_v$. □

Lemma 5.10. *The map* $\prod \overline{\mathrm{Mark}}_v \to \overline{\mathbb{M}}_{\mathrm{DMK}}$ *is continuous.*

Proof. Let T be the tree over which the product in the domain is taken. A variation of Lemma 4.3 for $\overline{\mathbb{M}}_{\mathrm{DMK}}$ shows that it suffices to show that for $\mathbf{x}$ in the (open) image of this map and $\mathscr{N}_{\mathbf{x},U}$ a neighborhood of $\mathbf{x}$ contained in this image, the preimage of $\mathscr{N}_{\mathbf{x},U}$ is a neighborhood of the preimage of $\mathbf{x}$, which is unique by Lemma 5.9. The underlying tree $T_{\mathcal{E}}$ of $\mathbf{x}$ is obtained from T by contracting some edges, and the restriction of the map

$$\prod_{v\in V(T)} \overline{\mathrm{Mark}}_v \to \overline{\mathbb{M}}_{\mathrm{DMK}}$$

to an appropriate open subspace factors as

$$\begin{tikzcd} \text{subspace of } \prod_T \overline{\mathrm{Mark}}_v \arrow[r] \arrow[dr] & \prod_{T_{\mathcal{E}}} \overline{\mathrm{Mark}}_v \arrow[d] \\ & \overline{\mathbb{M}}_{\mathrm{DMK}} \end{tikzcd}$$

where the horizontal map is composition of normalized representatives with $\odot$, which is continuous, yielding the result. $\square$

Proof of Proposition 5.2. By Lemmas 5.9 and 5.10, every point in $\overline{\mathbb{M}}_{\mathrm{DMK}}(S)$ has a neighborhood that looks like $\prod \overline{\mathrm{Mark}}_v$ for some tree T with vertices of the form $(\{s\},1)$ and $(S,1)$. For ease, forget these to got an a stable tree T'. By using a different $\mathrm{Aff}\,\mathbb{C}$ parameterization for the non-bivalent vertices, for example the one where c_0 is sent to 0 and c_1 to 1, it is easy to see that $\overline{\mathrm{Mark}}_v$ is a manifold with $2\mathrm{in}^{\mathrm{int}}(v)+\mathrm{in}^{\mathrm{ext}}(v)-2$ complex parameters. $\sum_v \mathrm{in}^{\mathrm{int}}(v)$ is the total number of internal edges, which is one less than the total number of vertices. $\sum_v \mathrm{in}^{\mathrm{ext}}(v)$ is $|S|$. Then $\prod \overline{\mathrm{Mark}}_v$ has total real dimension $2|S|-4$, as desired. $\square$

Proof of Proposition 5.3. Fix a point $\mathbf{x}$ in $\overline{\mathbb{M}}(S)$ with underlying tree T. Precompose and postcompose it with $(0,0)$ at every leaf and the root. This gives a continuous map to $\overline{\mathbb{M}}_{\mathrm{DMK}}(S)$, which is the retraction. Because it does not change a tree whose leaf and root vertices are bivalent and whose root vertex is decorated with $(0,0)$, it is the identity on $\overline{\mathbb{M}}_{\mathrm{DMK}}(S)$.

The homotopy is a map $H:\overline{\mathbb{M}}(S)\times[0,1]\to\overline{\mathbb{M}}(S)$; it is given by precomposing at each leaf and postcomposing at the root with $(0,t)$. This is clearly the retraction map at $t=0$ and the identity at $t=1$. It fixes $\overline{\mathbb{M}}_{\mathrm{DMK}}(S)$ as before. $\square$

Remark 1. The homotopy does not respect the operad structure in any intermediate stage. Its purpose is just to ensure that the induced map from the homology of $\overline{\mathbb{M}}_{\mathrm{DMK}}$ to that of $\overline{\mathbb{M}}$ is an isomorphism.

Now we will prove Proposition 5.4.

Definition 5.11. Fix a point $\mathbf{x}$ in $\overline{\mathbb{M}}_{\mathrm{DMK}}(S)$. Forget the vertices of the form $(\{s\},1)$ and $(S,1)$ in the underlying tree of $\mathbf{x}$. The decorations on each remaining vertex are configurations of points in the plane up to $\mathrm{Aff}\,\mathbb{C}$ indexed by the incoming edges of the vertex. By definition, such a configuration is a point of $\mathcal{M}(\mathrm{in}(v))$, and the product of these points is a point in $\overline{\mathcal{M}}$. This suffices to define

$$\Psi:\overline{\mathbb{M}}_{\mathrm{DMK}}\to\overline{\mathcal{M}}.$$

Proof that Ψ is an operad map. Composition in $\overline{\mathbb{M}}_{\mathrm{DMK}}$ grafts underlying trees, forgetting the two internal bivalent vertices that this creates, and preserves all other decorations. Ψ is compatible with this process, and thus is a map of operads. $\square$

Proof that Ψ is bijective. For an element in $\overline{\mathcal{M}}(S)$, insert a vertex on each external edge of the underlying tree. Keep the decorations on each old vertex, mark the new root with $(0,0)$, and each new leaf vertex with the unique possible marking. This constitutes an inverse map. $\square$

Proof that Ψ is continuous. Fix a point $\mathbf{x}$, which we will consider as being in both $\overline{\mathbb{M}}_{\mathrm{DMK}}(S)$ and $\overline{\mathcal{M}}(S)$. Let $\mathscr{N}_{\mathbf{x},U}$ be a neighborhood of $\mathbf{x}$ in $\overline{\mathcal{M}}$. We must show that there is a neighborhood $\mathscr{N}_{\mathbf{x},U'}$ in $\overline{\mathbb{M}}_{\mathrm{DMK}}$ contained in $\mathscr{N}_{\mathbf{x},U}$. Choose an ordering on S and fix the normalization with c_1 at 0, c_2 on the positive real axis and diameter $\frac{1}{2}$ for each decoration. So we have fixed a configuration in $\mathfrak{C}(\mathrm{in}(v))$ for each vertex of pairs $(c_e, 0)$.

Let $R > 1$ be a number so that the disk of radius R centered at 0 contains U_v for each vertex (in this normalization), and choose ϵ to be small enough so that the disk of radius $(R+1)\epsilon$ centered at c_e is contained in the complement of U_v for every edge e in $\mathrm{in}(v)$ for each vertex. Then let U_v' consist of configurations made of pairs (c_e', r_e') so that $|c_e' - c_e| < \epsilon$ and $|r_e'| < \epsilon$.

We must show that $\Psi(\mathscr{N}_{\mathbf{x},U'})$ is contained in $\mathscr{N}_{\mathbf{x},U}$. Any point in $\mathscr{N}_{\mathbf{x},U'}$ has a conformal subsurface at each vertex that looks like the disjoint union of a disk of radius one with some disks of radius $(R+1)\epsilon$ removed and annuli indexed by $\mathrm{in}(v)$ with outer radius Rr_e' and inner radius r_e' (in the case $r_e' > 0$. Composing the disk configuration into the corresponding annulus (which is part of the surface corresponding to the next vertex), we get a conformal subsurface containing U_v. There is an easy special case when $r_e' = 0$. $\square$

6. Homotopy pushouts of operads

This somewhat abstract section describes pushouts of operads $\mathcal{O}$ by monoids over $\mathcal{O}(1)$ in the very specific setting that $\mathcal{O}(1)$ is a group that acts freely on $\mathcal{O}(n)$ for $n > 1$, and shows that it is relatively easy to find homotopy models for such pushouts.

Definition 6.1. Let S be a finite set. The deleted S-cube category $\mathbf{Cu}_S$ has as its objects proper subsets of S and morphisms inclusions.

Definition 6.2. Let F and G be two functors from the discrete category S to $\mathcal{C}$, and let ξ be a natural transformation $F \to G$. Consider the functor from $\mathbf{Cu}_S$ to $\mathcal{C}$ which takes $P \subset S$ to

$$\prod_{s \in P} G(s) \times \prod_{s \notin P} F(s)$$

with maps from ξ. $\operatorname*{colim}_{\mathbf{Cu}_S}(F, G)$ is the colimit of this functor; it comes equipped with a map to $\prod_S G(s)$.

Lemma A.7, Corollary A.3, and Proposition A.6 imply the following lemma:

Lemma 6.3. *Let F and G be two functors from S to spaces and ξ be a natural cofibration. Then the induced map $\operatorname*{colim}_{\mathbf{Cu}_S}(F, G) \to \prod_S G(s)$ is a cofibration.*

Definition 6.4. Fix a collection $\mathcal{U}$, a space W, and a nonempty ordered finite set S. Let $\mathbb{T}_k(S)(\mathcal{U},W)$ be the space:

$$\coprod_{T\in\mathbb{T}_k(S)}\prod_{v\in V(T)}\mathcal{U}(E(v))\times\prod_{E(T)}W.$$

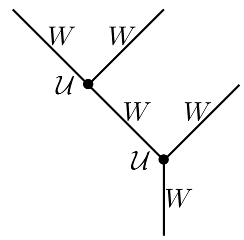

FIGURE 3. An picture of an element of $\mathbb{T}_k(S)(\mathcal{U},W)$.

Let K be a subspace of W; then let $\mathbb{T}_k(S)^K(\mathcal{U},W)$ be

$$\coprod_{T\in\mathbb{T}_k(S)}\prod_{v\in V(T)}\mathcal{U}(|E(v)|)\times\prod_{E^{\mathrm{ext}}(T)}W\times\operatorname*{colim}_{\mathbf{Cu}_{E^{\mathrm{int}}(T)}}(K,W),$$

the subspace of $\mathbb{T}_k(S)(\mathcal{U},W)$ consisting of points where at least one of the factors in the subproduct $\prod_{E^{\mathrm{int}}(T)}W$ is actually in K.

Definition 6.5. Fix $\mathcal{U}$, W, and K as in Definition 6.4. Suppose that for any internal edge e of a tree T (between vertices v and v'), there are *edge collapse maps* $\mathcal{U}(\mathrm{in}(v'))\times K\times\mathcal{U}(\mathrm{in}(v))\to\mathcal{U}(\mathrm{in}(\{v,v'\}))$, where $\{v,v'\}$ is the contraction vertex of T_e. Suppose these maps, along with canonical isomorphisms, extend the assignment $\mathbb{T}(S)\to\coprod\mathbb{T}_k(S)(\mathcal{U},K)$ to a functor F from trees to spaces. That is, the edge collapse maps satisfy associativity and identity constraints.

In this case, for $|S|>1$, define $\Omega_0(\mathcal{U},K,W)(S)$ to be $\mathbb{T}_1(S)(\mathcal{U},W)\cong W\times\mathcal{U}(S)\times W^S$. Suppose we have defined $\Omega_k(\mathcal{U},K,W)(S)$ so that it accepts a map from $\mathbb{T}_{k+1}(S)^K(\mathcal{U},W)$. Define $\Omega_{k+1}(\mathcal{U},K,W)(S)$ to be the pushout of the following diagram:

$$\begin{tikzcd}
\mathbb{T}_{k+1}(S)^K(\mathcal{U},W) \arrow[r] \arrow[d] & \Omega_k(\mathcal{U},K,W)(S)\ . \\
\mathbb{T}_{k+1}(S)(\mathcal{U},W) &
\end{tikzcd}$$

Then the morphisms of the functor F induce a map from $\mathbb{T}_{k+2}(S)^K(\mathcal{U},W)$ to $\Omega_{k+1}(\mathcal{U},K,W)(S)$, which is well-defined by functoriality (this also establishes the map in the base case).

Let $\Omega(\mathcal{U},K,W)(S)$ be the colimit of $\Omega_k(\mathcal{U},K,W)(S)$, which stabilizes at some finite k dependent on $|S|$.

If $|S|=1$, let $\Omega(\mathcal{U},K,W)(S)$ be $\mathbb{T}_0(S)(\mathcal{U},W)=W$.

Definition 6.6. A map ξ between two collections is a *cofibration of pointed collections* if it has the left lifting property with respect to all morphisms of pointed collections that are Serre fibrations on each finite set. We call a pointed collection *cofibrant as a pointed collection* if the morphism from I is a cofibration of pointed collections.

We will use Berger-Moerdijk's model category structure [BM02]:

Theorem 6.7. *There is a model category structure on topological operads where the fibrations (weak equivalences) are morphisms of operads which are Serre fibrations (induce isomorphisms of all homotopy groups) on each finite set.*

Theorem 6.8. *Let $\mathcal{O}$ be an operad with $\mathcal{O}(\emptyset)$ empty and for $|S| = 1$, $\mathcal{O}(S)$ an Abelian group. We will suppress S and refer to $\mathcal{O}(S)$ for $|S| = 1$ as K. Let $\mathcal{A}$ be a topological monoid and $K \to \mathcal{A}$ be a map of topological monoids. Assume that for $|S| > 1$, $\mathcal{O}(S)$ is a free $K - K^S$ bimodule and that the quotient splits so that $K \times \left(K\backslash\mathcal{O}(S)/K^S\right) \times K^S \cong \mathcal{O}(S)$ as topological spaces with actions of K and K^S (the action of $\mathbb{S}_S$ on the quotient and on the factors of K^S may be twisted by some induced action on the K factors, including the single factor on the left). Define $\mathcal{U}$ as the collection with $\mathcal{U}(S) = K\backslash\mathcal{O}(S)/K^S$. We will view $\mathcal{U}(S)$ as a subspace of $\mathcal{O}(S)$, using the identity in K.*

Then $\Omega(\mathcal{U}, K, \mathcal{A})$ is the pushout of the diagram $\mathcal{O} \leftarrow K \to \mathcal{A}$ in the category of operads.

Proof. First, $\Omega(\mathcal{U}, K, \mathcal{A})$ must be given an operad structure. We begin by describing the symmetric group action on $\Omega(\mathcal{U}, K, \mathcal{A})(S)$. Consider a point $\mathbf{x} = \prod \mathbf{x}_v \times \prod \mathbf{x}_e$ in $\Omega(\mathcal{U}, K, \mathcal{A})(S)$ with underlying tree T, and a permutation $\sigma \in \mathbb{S}_S$. First, $\mathbb{S}_S$ acts on S-trees naturally, so $\sigma(\mathbf{x})$ will have underlying tree σT.

The incoming edges of a vertex are pairs (P, m) where P is a subset of S. σ induces an action on finite the set of finite subsets of S which induces a bijection between the incoming edges of a vertex in T to and the incoming edges of the corresponding vertex in σT.

An element $\mathbf{x}_v$ of $\mathcal{U}(\mathrm{in}(v))$, viewed in $\mathcal{O}(\mathrm{in}(v))$, is taken by this action to an element of $\mathcal{O}(\mathrm{in}(v))$ of the form

$$\mathbf{y}_v \times \mathbf{x}'_v \times \prod_{e \in \mathrm{in}(v)} \mathbf{y}_e$$

for some $\mathbf{x}'_v \in \mathcal{U}(\mathrm{in}(v))$ and $\mathbf{y}_v, \mathbf{y}_e \in K$. If $\mathbf{x}_v$ is the label of v, let $\mathbf{x}'_v$ be the label on $\sigma(v)$. For a fixed edge e labeled by $\mathbf{x}_e$, letting v denote the vertex with e as its outgoing edge, label $\sigma(e)$ with $\mathbf{y}_e \mathbf{x}_e \mathbf{y}_v^{-1}$. This is the twisting of the induced action mentioned above.

Now we describe the structure maps. Grafting of trees with standard relabelling gives maps

$$\mathbb{T}_k(S)(\mathcal{U}, \mathcal{A}) \times \mathbb{T}_{k'}(S')(\mathcal{U}, \mathcal{A}) \to \mathbb{T}_{k+k'}(S\backslash\{s\} \sqcup S')(\mathcal{U}, \mathcal{A}),$$

which passes to the colimit $\Omega(\mathcal{U}, K, \mathcal{A})$. The tree with no vertices labelled with the identity in $\mathcal{A}$ is the identity for this composition.

For $|S| = 1$, $\Omega(\mathcal{U}, K, \mathcal{A})(S)$ is $\mathcal{A}$, and for $|S| > 1$, there is a map from $\mathcal{O}(S) \cong \mathbb{T}_1(S)(\mathcal{U}, K$ to $\Omega_0(\mathcal{U}, K, \mathcal{A})(S) = \mathbb{T}_1(S)(\mathcal{U}, W)$. Clearly, the images of $\mathcal{O}$ and $\mathcal{A}$ generate all of $\Omega(\mathcal{U}, K, \mathcal{A})$, so for a fixed operad $\mathcal{P}$ the map from $\mathrm{Hom}(\Omega(\mathcal{U}, K, \mathcal{A}), \mathcal{P})$ to $\mathrm{Hom}(\mathcal{O}, \mathcal{P}) \times_{\mathrm{Hom}(K, \mathcal{P})} \mathrm{Hom}(\mathcal{A}, \mathcal{P})$ is injective. To see that it is surjective, let $\Phi_{\mathcal{O}}$ and $\Phi_{\mathcal{A}}$ be compatible maps $\mathcal{O} \to \mathcal{P}$ and $\mathcal{A} \to \mathcal{P}$. These define $\mathbb{S}_S$-equivariant maps $\mathbb{T}_k(S)(\mathcal{U}, \mathcal{A}) \to \mathcal{P}(S)$ compatible with the grafting maps. $\square$

Proposition 6.9. *Let $\mathcal{O}$ and $\mathcal{A}$ be a pair as in Theorem 6.8, let $\mathcal{O}'$ and $\mathcal{A}'$ be another such pair, and suppose there are maps between them forming the diagram*

$$\begin{tikzcd} \mathcal{O} \arrow[d] & K \arrow[l] \arrow[r] \arrow[d] & \mathcal{A} \arrow[d] \\ \mathcal{O}' & K' \arrow[l] \arrow[r] & \mathcal{A}' \end{tikzcd}$$

so that $K \to K'$ is an isomorphism (which we shall suppress), $\mathcal{A} \to \mathcal{A}'$ is a weak equivalence of monoids, and $\mathcal{O} \to \mathcal{O}'$ is a weak equivalence of operads so that $\mathcal{O}(S) \to \mathcal{O}'(S)$ is $K - K^S$-equivariant.

Assume that K and $\mathcal{A}$, and $\mathcal{U}(S) = K\backslash\mathcal{O}(S)/K^S$ are cofibrant spaces, and likewise for $\mathcal{A}'$ and $\mathcal{O}'$, and that $K \to \mathcal{A}$ and $K \to \mathcal{A}'$ are cofibrations of spaces.

Then the induced map of pushouts is a weak equivalence.

Proof. The long exact sequence of homotopy groups and the five lemma imply a weak equivalence from $\mathcal{U}(S) \to \mathcal{U}'(S)$. Since products preserve weak equivalences of spaces, we have a weak equivalence $\Omega_0(\mathcal{U}, K, \mathcal{A})(S) \to \Omega_0(\mathcal{U}', K, \mathcal{A}')(S)$ for $|S| > 1$. Proposition A.6 and the fact that coproducts also preserve weak equivalences of cofibrant spaces imply that we have weak equivalences

$$\mathbb{T}_k(S)^K(\mathcal{O}, \mathcal{A}) \to \mathbb{T}_k(S)^K(\mathcal{O}', \mathcal{A}')$$

and

$$\mathbb{T}_k(S)(\mathcal{O}, \mathcal{A}) \to \mathbb{T}_k(S)(\mathcal{O}', \mathcal{A}').$$

$\mathbb{T}_k(S)^K(\mathcal{O}, \mathcal{A})$ and $\mathbb{T}_k(S)^K(\mathcal{O}', \mathcal{A}')$ are cofibrant spaces and the maps to $\mathbb{T}_k(S)(\mathcal{O}, \mathcal{A})$ and $\mathbb{T}_k(S)(\mathcal{O}', \mathcal{A}')$ are cofibrations by Corollary A.3 and Lemma 6.3. Then by induction, if

$$\Omega_k(\mathcal{U}, K, \mathcal{A})(S) \to \Omega_k(\mathcal{U}', K, \mathcal{A}')(S)$$

is a weak equivalence of cofibrant spaces, so is

$$\Omega_{k+1}(\mathcal{U}, K, \mathcal{A})(S) \to \Omega_{k+1}(\mathcal{U}', K, \mathcal{A}')(S),$$

using the nearly direct category structure of the pushout diagram (see Proposition A.6).

Then, by the same proposition, $\Omega_\bullet(\mathcal{U}, K, \mathcal{A})(S) \to \Omega_\bullet(\mathcal{U}', K, \mathcal{A}')(S)$ is a weak equivalence of cofibrant telescopes so the weak equivalence passes to the colimit. $\square$

Theorem 6.10. *If $\mathcal{O} \leftarrow \mathcal{P} \to \mathcal{Q}$ is a diagram of operads which are cofibrant as pointed collections, $\mathcal{P} \to \mathcal{O}$ is a cofibration of operads, and $\mathcal{P} \to \mathcal{Q}$ is a weak equivalence, then the induced map $\mathcal{O} \to \mathcal{O} \sqcup_{\mathcal{P}} \mathcal{Q}$ is a weak equivalence of operads and $\mathcal{Q} \to \mathcal{O} \sqcup_{\mathcal{P}} \mathcal{Q}$ is a cofibration of pointed collections.*

This is parts of Theorem 3.2 and Proposition 3.6 of [Spi01].

The following construction is classical.

Theorem 6.11 ([BV73, BM06])**.** *There exists a functor $\mathbb{W}$ on operads $\mathcal{O}$ which are cofibrant as pointed collections so that $\mathbb{W}\mathcal{O}$ is cofibrant as both an operad and a pointed collection, and a natural weak equivalence $\mathbb{W} \to \mathrm{Id}$. $\mathbb{W}$ takes cofibrations of pointed collections to maps that are cofibrations of operads.*

Lemma 6.12. *There exists an operad $\mathbb{E}$ satisfying the following properties:*

1. $\mathbb{E}(S)$ *is a point if* $|S| = 1$
2. $\mathbb{E}(S)$ *is contractible*

(3) $\mathbb{E}(S)$ *is cofibrant as a pointed collection.*

Proof. There is such an operad where $\mathbb{E}(S)$ is (some functorial version of) the total space of the universal bundle over the classifying space of $\mathbb{S}_S$, with structure maps induced by the concomitant maps among the various $\mathbb{S}_S$. There is a standard choice satisfying the first condition. $\square$

Lemma 6.13. *Let $\mathcal{O}$ be an operad so that the unit map is a cofibration of spaces. Consider the operad $\mathcal{O} \times \mathbb{E}$ with $(\mathcal{O} \times \mathbb{E})(S) = \mathcal{O}(S) \times \mathbb{E}(S)$, using the product of the structure maps of $\mathcal{O}$ and $\mathbb{E}$. Then the projection map $\mathcal{O} \times \mathbb{E} \to \mathcal{O}$ is a weak equivalence and $\mathcal{O} \times \mathbb{E}$ is cofibrant as a pointed collection.*

Proof. The weak equivalence is obvious. The (not necessarily equivariant) morphism space from $\mathcal{O}(S)$ into a fibration with an action of $\mathbb{S}_S$ form a fibration of spaces with $\mathbb{S}_S$ action (by conjugation). $\mathbb{E}(S)$ has the left lifting property with respect to this fibration. Because $\mathbb{S}_S$-spaces form a Cartesian closed category, we get the desired lifting property on the product operad. There is a special argument when $|S| = 1$. $\square$

Proposition 6.14. *Let $\mathcal{O} \leftarrow \mathcal{P} \to \mathcal{Q}$ be a diagram of operads, so that all objects are cofibrant pointed collections and all maps are cofibrations of pointed collections. There is an operad $\widehat{\mathcal{Q}}$ equipped with maps $\mathcal{P} \to \widehat{\mathcal{Q}} \to \mathcal{Q}$ which factor $\mathcal{P} \to \mathcal{Q}$ into a cofibration of pointed collections followed by a weak equivalence and so that $\mathcal{O} \leftarrow \mathcal{P} \to \widehat{\mathcal{Q}}$ is a model for the homotopy pushout of $\mathcal{O} \leftarrow \mathcal{P} \to \mathcal{Q}$.*

Proof. Let $\widehat{\mathcal{Q}}$ be the pushout of the diagram $\mathcal{P} \leftarrow \mathbb{W}\mathcal{P} \to \mathbb{W}\mathcal{Q}$. This clearly comes with maps $\mathcal{P} \to \widehat{\mathcal{Q}} \to \mathcal{Q}$ which factor $\mathcal{P} \to \mathcal{Q}$. By Theorem 6.10, the map $\mathcal{P} \to \widehat{\mathcal{Q}}$ is a cofibration of pointed collections. By Theorems 6.10 and 6.11, the map $\widehat{\mathcal{Q}} \to \mathcal{Q}$ is a weak equivalence.

The pushout of $\mathcal{O} \leftarrow \mathcal{P} \to \widehat{\mathcal{Q}}$ is canonically isomorphic to the pushout of $\mathcal{O} \leftarrow \mathbb{W}\mathcal{P} \to \mathbb{W}\mathcal{Q}$, and it suffices to show that this latter diagram is a model for the homotopy pushout.

By Theorem 6.11, $\mathbb{W}\mathcal{O} \leftarrow \mathbb{W}\mathcal{P} \to \mathbb{W}\mathcal{Q}$ is a literal model for the homotopy pushout, with realization $\mathcal{R}$. By Theorems 6.10 and 6.11, the induced map from $\mathcal{R}$ to the pushout of the diagram $\mathcal{O} \leftarrow \mathbb{W}\mathcal{O} \to \mathcal{R}$ is a weak equivalence. But $\mathcal{O} \leftarrow \mathbb{W}\mathcal{O} \to \mathcal{R}$ is canonically isomorphic to $\mathcal{O} \leftarrow \mathbb{W}\mathcal{P} \to \mathbb{W}\mathcal{Q}$, and the weak equivalence from $\mathcal{R}$ to the pushout of this diagram arises from the map of diagrams from $\mathbb{W}\mathcal{O} \leftarrow \mathbb{W}\mathcal{P} \to \mathbb{W}\mathcal{Q}$ to $\mathcal{O} \leftarrow \mathbb{W}\mathcal{P} \to \mathbb{W}\mathcal{Q}$. $\square$

Now we can combine Propositions 6.9 and 6.14 to achieve the main theorem of this section.

Theorem 6.15. *Let $\mathcal{O}$ and $\mathcal{A}$ be a pair as in Theorem 6.8. Assume that $K \to \mathcal{A}$ is a cofibration between cofibrant pointed spaces (rather than just a cofibration between cofibrant spaces) Then the diagram $\mathcal{O} \leftarrow K \to \mathcal{A}$ is a model for its homotopy pushout.*

Proof. By Lemma 6.13 and Proposition 6.14, the diagram $\mathcal{O} \times \mathbb{E} \leftarrow K \to \widehat{\mathcal{A}}$ is a model for the homotopy pushout (K and $\mathcal{A}$ are trivially cofibrant as pointed collections). By Proposition 6.9 the induced map of pushouts from its pushout to the pushout of $\mathcal{O} \leftarrow K \to \mathcal{A}$ is a weak equivalence. $\square$

7. Homotopy trivializing the circle in the framed little disks

In this section, we connect the two pushouts that we have constructed to prove our main result, Theorem 1.1

Definition 7.1. $\overline{\mathbb{M}}_{\bigcirc}$ is the pushout of the diagram $FLD_{\bigcirc} \leftarrow S^1 \to An^{\mathrm{triv}}$.

Lemma 7.2. *$\overline{\mathbb{M}}_{\bigcirc}$ is a model of the homotopy pushout of $FLD_{\bigcirc} \leftarrow S^1 \to I$.*

Proof. This follows from Theorem 6.15. The substitution of I for An^{triv} makes no difference as there is a map $An^{\mathrm{triv}} \to I$ making all diagrams commute. $\square$

Theorem 7.3. *The natural map $\overline{\mathbb{M}}_{\bigcirc} \to \overline{\mathbb{M}}$ is a weak equivalence.*

Theorem 1.1 now follows.

In order to prove Theorem 7.3, it will suffice to show that its postcomposition $\psi : \overline{\mathbb{M}}_{\bigcirc} \to \overline{\mathbb{M}} \to \overline{\mathcal{M}}$ is a weak equivalence.

Lemma 7.4. *[Folklore] Let $\varphi : W \to X$ be a map and $\mathbb{O}$ an open cover of X which is closed under finite intersections. If $\varphi : \varphi^{-1}(U) \to U$ is a weak equivalence for all $U \in \mathbb{O}$ then φ is a weak equivalence.*

There is a proof of this lemma in [May90].

Definition 7.5. Let W and X be spaces. A *deformation retraction over X* is the data of a deformation retraction with projection map π from W onto X with specified section and homotopy H so that the following diagram commutes:

$$\begin{tikzcd} I \times W \arrow[r, "h"] \arrow[d, "\mathrm{proj}_2"'] & W \arrow[d, "\pi"] \\ W \arrow[r, "\pi"'] & X \end{tikzcd}$$

Lemma 7.6. *Let $\varphi : W \to X$ be the retraction map of a deformation retraction over X. Then for any subset U of X, the restriction of φ to $\varphi^{-1}(U)$ is a weak equivalence.*

Proof. The homotopy of the deformation retraction restricts to $\varphi^{-1}(U)$. $\square$

Definition 7.7. Let $\mathbf{x}$ be a point in $\overline{\mathcal{M}}(S)$ with underlying stable tree T, and let S' be a subset of S. Let v be an vertex of T containing S'. Choose an $\mathrm{Aff}\,\mathbb{C}$ representative $\mathbf{x}_v$ of the decoration of v. Each marked point in $\mathbf{x}_v$ is indexed by some edge e. The *center of mass $L_{v,S'}$* of S' in $\mathbf{x}_v$ is a point in $\mathbb{C}$ defined to be the weighted average of the marked points of $\mathbf{x}_v$, where the weight of the marking indexed by e is $|e \cap S'|$. This is $\mathrm{Aff}\,\mathbb{C}$-equivariant.

Definition 7.8. A *stable subset* of the finite set S is a proper subset with at least two elements.

Definition 7.9. Let $\mathscr{S} = \{S_i\}$ be a set of stable subsets of S. The *$\mathscr{S}$-stratum* of $\overline{\mathcal{M}}(S)$ consists of points whose underlying tree has an edge of the form S_i for every i. The *codimension* of a stratum is $|\mathscr{S}|$. The *open $\mathscr{S}$-stratum* is the complement of all strata of higher codimension in the $\mathscr{S}$-stratum.

Definition 7.10. Let $\mathscr{S} = \{S_i\}$ be a set of stable subsets of S. The open $\mathscr{S}$-set in $\overline{\mathcal{M}}(S)$ consists of points so that for each edge e of a point's underlying tree and each stable subset S_i, the intersection $e \cap S_i$ is either empty, e, or S_i.

Let $\mathbf{x}$ be in the $\mathscr{S}$-set, with underlying tree T. Suppose we have normalized the decorations $\mathbf{x}_v$ on each vertex of T with respect to $\mathrm{Aff}\,\mathbb{C}$ in some way. A *simultaneous division of* $\mathbf{x}$ along $\mathscr{S}$ is a set of radii $r_{v,i}$ for each pair consisting of a vertex v of T and an index of $\mathscr{S}$ such that S_i is a union of incoming edges $\{e_{i,j}\}$ of v. These radii should satisfy the following:

(1) the circle in $\mathbb{C}$ centered at L_{v,S_i} separates those marked points in $\mathrm{in}(v)$ corresponding to the edges $e_{i,j}$ from all other marked points and ∞, and
(2) for fixed v, any two such circles are disjoint.

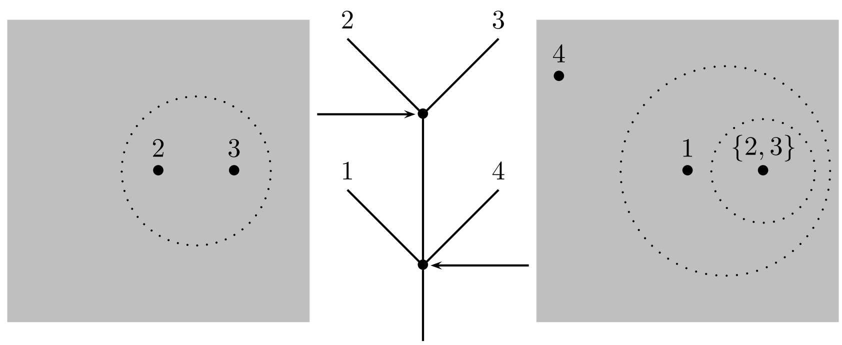

FIGURE 4. A simultaneous division of a point in $\overline{\mathcal{M}}(\{1,2,3,4\})$ along $\{\{2,3\},\{1,2,3\}\}$

Definition 7.11. Let T be a S-tree. The T-neighborhood $\mathscr{N}_T$ in $\overline{\mathcal{M}}(S)$ consists of points which can be simultaneously divided along the stable subsets $E^{\mathrm{int}}(T)$, and do not belong to codimension one strata except possibly the $\{e\}$-strata, for $e \in E^{\mathrm{int}}(T)$.

Lemma 7.12. *The T-neighborhoods are open and cover $\overline{\mathcal{M}}$.*

Proof. Having a simultaneous division is an open condition, and the T-stratum is contained in the T-neighborhood. $\square$

Definition 7.13. Let T be a stable S-tree. Consider

$$\prod_{v\in V(T)} FLD_{\bigcirc}^{\mathrm{norm}}(E(v)) \times \prod_{E^{\mathrm{int}}(T)} An^{\mathrm{triv}} \times \prod_{E^{\mathrm{ext}}(T)} \{(0,0)\}$$

as a subspace of $\mathbb{T}_{|V(T)|}(S)(FLD_{\bigcirc}^{\mathrm{norm}}, An^{\mathrm{triv}})$. Then it comes with a map to $\overline{\mathbb{M}}_{\bigcirc}$, which is isomorphic to $\Omega(FLD_{\bigcirc}^{\mathrm{norm}}, S^1, An^{\mathrm{triv}})$ by Theorem 6.8, and thus to $\overline{\mathcal{M}}$. We will call this map τ. Let $\overline{\mathfrak{m}}_T$ be the preimage under τ of the T-neighborhood, so that the following diagram commutes:

$$\begin{tikzcd} \overline{\mathfrak{m}}_T \arrow[r] \arrow[rr, bend left, "\tau"] & \overline{\mathbb{M}}_{\bigcirc} \arrow[r, "\psi"] & \overline{\mathcal{M}} \end{tikzcd}$$

We will use $\overline{\mathfrak{m}}_T$ as an intermediate space to show that $\psi^{-1}(\mathscr{N}_T) \to \mathscr{N}_T$ is a weak equivalence.

Lemma 7.14. *$\tau : \overline{\mathfrak{m}}_T \to \mathscr{N}_T$ is a deformation retraction over $\mathscr{N}_T$.*

Construction of a section Γ *of* τ. Let $\mathbf{x}$ be a point in $\mathscr{N}_T$ with underlying tree $T_{\mathbf{x}}$. We will define a point $\Gamma(\mathbf{x})$ over it in $\overline{\mathfrak{m}}_T$. We must specify a decoration in $FLD^{\mathrm{norm}}_{\bigcirc}$ for each vertex and An^{triv} for each edge.

Since $\mathbf{x}$ can be simultaneously divided along $E^{\mathrm{int}}(T)$, the tree T can be contracted to the tree $T_{\mathbf{x}}$. Any edge of T which is not so contracted is decorated by $0 \in An^{\mathrm{triv}}$. Then it will suffice to consider the case when $T_{\mathbf{x}}$ has only one vertex; in the general case we can use the same procedure for each vertex of $T_{\mathbf{x}}$.

So in the case where $T_{\mathbf{x}}$ has one vertex, we can consider $\mathbf{x}$ in $\mathrm{Emb}(S, \mathbb{C})/\operatorname{Aff}\mathbb{C}$, so a set of S-indexed disjoint points up to $\operatorname{Aff}\mathbb{C}$. Modify this configuration to a new set of points by adding the centers of mass L_e of each edge e of T. These may coincide with one another and with the points indexed by S.

Choose an $\operatorname{Aff}\mathbb{C}$-normalization for this set. We will define a function $r_{\mathrm{inf}} : E(T) \to [0, \infty)$. If e is a leaf, then $r_{\mathrm{inf}}(e) = 0$. Assume that r_{inf} has been defined on every edge e' which is a subset of e. Then $r_{\mathrm{inf}}(e)$ is the infimum radius so that the disk centered at L_e of radius $r_{\mathrm{inf}}(e)$ contains the disks centered at $L_{e'}$ of radius $r_{\mathrm{inf}}(e')$. This set is nonempty because $\mathbf{x}$ is in the T-neighborhood.

Next, we will define another function $r_{\mathrm{sup}} : E(T) \to (0, \infty]$. For each edge, once r_{inf} and r_{sup} are both defined, let $r_{\mathrm{har}}(e)$ and $r_{\mathrm{ari}}(e)$ be, respectively, the harmonic and arithmetic mean of $r_{\mathrm{inf}}(e)$ and $r_{\mathrm{sup}}(e)$.

For e the root of T, let $r_{\mathrm{sup}}(e) = \infty$. Now assume r_{sup} is defined for an edge e, and that the edges $e_1, \ldots, e_m$ are the edges of T which have e as their output. For i in $1, \ldots, m$, let

$$\Delta_i = \min\left\{|L_{e_i} - L_e| - r_{\mathrm{inf}}(e_i) + r_{\mathrm{har}}(e), \{|L_{e_i} - L_{e_j}| - r_{\mathrm{inf}}(e_i) - r_{\mathrm{inf}}(e_j)\}_{i \neq j}\right\}$$

It is easy to see that $\Delta_i > 0$. Let

$$r_{\mathrm{sup}}(e_i) = r_{\mathrm{inf}}(e_i) + \frac{\Delta_i}{2}.$$

Now consider the pairs

$$\left(\frac{L_{e_i} - L_e}{r_{\mathrm{har}}(e)}, \frac{r_{\mathrm{ari}}(e_i)}{r_{\mathrm{har}}(e)}\right).$$

The set of these for e_i an incoming edge of $v \in V(T)$ is an element $\mathbf{x}'_v$ in $\mathfrak{C}(\mathrm{in}(v))$. The choice depends on the normalization, but only up to rotation. There is a rotation θ_v so that $(0, e^{i\theta_v}) \odot \mathbf{x}'_v$ is in $FLD^{\mathrm{norm}}_{\bigcirc}(\mathrm{in}(v))$. Let this be the decoration on the vertex v. Let the decoration on the edge e be

$$\frac{r_{\mathrm{har}}(e)}{r_{\mathrm{ari}}(e)} e^{-i\theta_v}.$$

Translation does not affect any of the radii or a difference in centers of mass. Rotation only affects centers of mass, and is accounted for by θ_v. Dilation acts on all radii and each center of mass in the same manner, so these formulae do not depend on the $\operatorname{Aff}\mathbb{C}$-representative used to generate them.

If we compose the decorations of $\Gamma(\mathbf{x})$ using the map $\odot$, we eventually arrive at a set of points

$$\frac{L_\ell - L_*}{r_{\mathrm{har}}(*)}$$

indexed by the leaves ℓ of T, where $*$ is the root of T. The set of these is conformally equivalent to the set $\{L_\ell\}$ by the action of $(L_*, r_{\mathrm{har}}(*)) \in \operatorname{Aff}\mathbb{C}$. Each L_ℓ is the center of mass of one point, and so this shows that Γ is a section of τ.

To see continuity within an open stratum, consider that L_e varies continuously as one moves around an open stratum; then $r_{\mathrm{inf}}(e)$ (in a fixed $\mathrm{Aff}\,\mathbb{C}$ representative) can be realized as the distance from L_e to the complement of some disks which depend on other L_e and earlier $r_{\mathrm{inf}}(e)$. Then r_{inf} varies continuously by induction. The same is true for r_{sup}, so r_{har} and r_{ari} also vary continuously.

Now suppose the sequence $\mathbf{x}_k$ in an open stratum approaches a degeneration $\mathbf{x}$ in a higher codimension stratum in $\overline{\mathcal{M}}$. If e is an edge that does not correspond to a node in $\mathbf{x}$, then by appropriately renormalizing, the decoration on e in $\mathbf{x}_k$ approaches that in $\mathbf{x}$ as in the previous paragraph. If, on the other hand, e corresponds to a node in $\mathbf{x}$, then let e_0 and e_1 be two incoming edges of e. Normalize by placing L_{e_0} and L_{e_1} at 0 and 1, respectively; $r_{\mathrm{inf}}(e)$ will stay bounded and bounded away from zero, while $r_{\mathrm{sup}}(e)$ will blow up. Then $\frac{r_{\mathrm{har}}(e)}{r_{\mathrm{ari}}(e)}$ will approach 0, and we will achieve the desired edge decoration in the limit.

Switching normalizations, normalize $\mathbf{x}_k$ to make $r_{\mathrm{har}}(e) = 1$ (this can be done unless e is a leaf), put L_e at 0, and rotate so that $\theta_e = 0$. In this normalization, the decoration on the vertex e consists of pairs $(L_{e_i}, r_{\mathrm{ari}}(e_i))$. In the limit, if e corresponds to a node, then we must be careful; it is involved in two trees and has different radius values in each. In the upper tree, the one pertinent to the case we are considering, we have finite $r_{\mathrm{inf}}(e)$, so we can use the same normalization, and we get the limits of the sets of pairs. That is, L_{e_i} is insensitive to whether points coincide in the limit, and $r_{\mathrm{ari}}(e_i)$ depends only on $r_{\mathrm{sup}}(e)$, which approaches ∞ as $\mathbf{x}_k \to \mathbf{x}$, and $r_{\mathrm{inf}}(e_j)$, which approaches 0 in those cases where e_j also corresponds to a node in $\mathbf{x}$. $\square$

Definition 7.15 (Construction of a homotopy for Lemma 7.14)**.** As in the construction of Γ, we will define the homotopy explicitly only in the top stratum. In other strata, where some edges must be decorated with $0 \in \overline{\mathbb{M}}$, we will use multiple copies of the homotopy for various subtrees glued together along these edges.

We can describe the portion of $\overline{\mathfrak{m}}_T$ above the top stratum in a different presentation. Up to $\mathrm{Aff}\,\mathbb{C}$, its image in $\overline{\mathcal{M}}$ is a set of disjoint points in $\mathbb{C}$ indexed by the leaves of T which is realized by composition of the various vertex and edge decorations. Therefore, we can realize a point of $\overline{\mathfrak{m}}_T$ as a set of disjoint points and circles; for an edge decorated by the point r, we draw circles δ_e^1 and δ_e^r (we will refer to them collectively as δ_e) corresponding to the image of the circles of radius 1 and $|r|$. At the leaves, the inner "circle" has radius 0 and coincides with the marked point, and at the root the outer "circle" has radius ∞. Finally, the location of the marked point on each boundary circle is determined by the normalizations involved. We have said enough to justify the following description:

Lemma 7.16. *Let S be a finite set, and fix a S-tree T. The top stratum $\overline{\mathfrak{m}}_T^{\mathrm{Top}}$ of $\overline{\mathfrak{m}}_T$ is homeomorphic to the subset of $\mathfrak{C}(E(T)\backslash\{root\}) \times \mathfrak{C}(E(T))$ consisting of configurations $\prod(c_e, r_e) \times \prod(\hat{c}_e, \hat{r}_e)$ so that*

(1) *For a fixed edge, the disk centered at c_e of radius r_e contains the disk centered at $\hat{c}_e$ of radius r_e*
(2) *All radii are real*[2]

[2]This does not mean that the marked points of the various configurations are all in the positive real direction from the corresponding centers. Rather, because the normalizations determine the arguments of the marked points, we lose no information by forgetting them, and this choice is a convenient one.

(3) *If e contains e' then the disk centered at c_e of radius $\hat{r}_e$ contains the disk centered at $c_{e'}$ of radius $r_{e'}$*
(4) *e is a leaf if and only if $\hat{r}_e = 0$.*

We will now describe an $\operatorname{Aff}\mathbb{C}$-equivariant homotopy over the top stratum of $\overline{\mathcal{M}}$ from the identity to Γ. The beginning and end of the homotopy, at any point in the domain of the homotopy, have the same underlying configuration of points, which we arbitrarily normalize once and for all. They also have the same nesting pattern for the various circles; that is, for a particular point $\mathbf{x}$, if the disk bounded by one such circle is contained in the disk bounded by another at time 0, then it is also so contained at time 1 (with some obvious exceptions if two disks coincide). In particular, since every circle contains some marked points, which are in the same position at the beginning and end of the homotopy, the open disks bounded by $\delta_e(\mathbf{x})$ and $\delta_e(\Gamma\lambda\mathbf{x})$ have nonempty intersection.

Focussing on one particular circle δ_e, we have a specified center and radius at time 0 and 1, c_0, c_1, r_0, and r_1. We will extend these to c_t and r_t for all $t \in I$.

Consider the subset Λ of points $(c_0, r_0, c_1, r_1) \subset \mathbb{C}^2 \times \mathbb{R}_+^2$ so that $|c_0 - c_1| < r_0 + r_1$. We call such pairs of disks *properly overlapping*. We will define a map $h : \Lambda \times I \to \mathbb{C} \times \mathbb{R}_+$, $h = (c_t, r_t)$.

First define
$$\theta(c_0, r_0, c_1, r_1) = \arccos\left(\frac{r_0^2 + r_1^2 - |c_0 - c_1|^2}{2r_0r_1}\right).$$
Because of the conditions, the quantity inside the arccosine is strictly greater than -1, so θ (we will suppress the arguments) is either in $[0, \pi)$ or of the form ai for some positive number a. arccos is a bijection between this domain and range.

Now, for $\theta \neq 0$, define
$$c_t = \frac{c_1 r_0 \sin(t\theta) + c_0 r_1 \sin((1-t)\theta)}{r_0 \sin(t\theta) + r_1 \sin((1-t)\theta)}, \qquad r_t = \frac{r_0 r_1 \sin(\theta)}{r_0 \sin(t\theta) + r_1 \sin((1-t)\theta)};$$
If $\theta = 0$, let
$$c_t = \frac{c_1 r_0 t + c_0 r_1 (1-t)}{r_0 t + r_1(1-t)}, \qquad r_t = \frac{r_0 r_1}{r_0 t + r_1 (1-t)}.$$
This is easily continuous by the squeeze theorem.

Definition 7.17. Let Θ be the subset of $\overline{\mathfrak{m}}_T^{\mathrm{Top}} \times \overline{\mathfrak{m}}_T^{\mathrm{Top}}$ consisting of tuples
$$\prod(c_{e,0}, r_{e,0}) \times \prod(\hat{c}_{e,0}, \hat{r}_{e,0}) \times \prod(c_{e,1}, r_{e,1}) \times \prod(\hat{c}_{e,1}, \hat{r}_{e,1})$$
so that

(1) $(c_{e,0}, r_{e,0}, c_{e,1}, r_{e,1})$ is in Λ,
(2) likewise, if e is not a leaf, $(\hat{c}_{e,0}, \hat{r}_{e,0}, \hat{c}_{e,1}, \hat{r}_{e,1})$ is in Λ, and
(3) for λ a leaf, $c_{\lambda,0} = c_{\lambda,1}$.

Define a homotopy $H : \Theta \times I \to (\mathbb{C} \times \mathbb{R}_+)^{E(T)\setminus\{\mathrm{root}\}} \times (\mathbb{C} \times \mathbb{R}_+)^{E(T)}$ by taking $\{c_{e,i}, r_{e,i}; \hat{c}_{e,i}, \hat{r}_{e,i}\}, t$ to
$$\{h_t(c_{e,0}, r_{e,0}, c_{e,1}, r_{e,1}); h_t(\hat{c}_{e,0}, \hat{r}_{e,0}, \hat{c}_{e,1}, \hat{r}_{e,1})\}$$
where $h_t(\hat{c}_{\lambda,0}, \hat{r}_{\lambda,0}, \hat{c}_{\lambda,1}, \hat{r}_{\lambda,1})$, which was not defined, is interpreted as $(\hat{c}_{\lambda,0}, \hat{r}_{\lambda,i})$, which is independent of i.

The continuity of h implies the continuity of H on Θ.

Proposition 7.18. *The image of the homotopy H on Θ is in $\overline{\mathfrak{m}}_T^{\mathrm{Top}}$.*

The proof of this proposition is elementary, albeit intricate, and will be deferred to Appendix D.

Proof of Lemma 7.14. We have constructed a section, Γ. Because this is a section, for $(\mathbf{x}_i)$ in the open stratum of $\overline{\mathfrak{m}}_T$, $(\mathbf{x}_i, (\Gamma\tau\mathbf{x})_i)$ is in Θ so that $(\mathrm{Id}, \Gamma\tau)$ is a map from $\overline{\mathfrak{m}}_T$ to Θ. Then $H \circ (\mathrm{Id}, \Gamma\tau)$ is a homotopy between Id and $\Gamma\tau$ on the open stratum of $\overline{\mathfrak{m}}_T$ over $\mathscr{N}_T$.

It only remains to show that this homotopy has the appropriate limiting behavior as one approaches a lower stratum. This follows from the facts that Γ has the appropriate behavior as described above and that h is Aff $\mathbb{C}$-equivariant, with the simultaneous action on both factors of Λ. $\square$

Lemma 7.19. *Let U be the intersection of a finite set $\{\mathscr{N}_{T_i}\}$. Then for all i, the map from the preimage of U in $\overline{\mathfrak{m}}_{T_i}$ to the preimage of U in $\overline{\mathbb{M}}_{\bigcirc}$ is a weak equivalence.*

Proof. We will eventually follow the proof of Lemma 7.14, but cannot do so immediately because it is not obvious how to pick a section without modifying the situation. This is because of the equivalence relation that equates different configurations with "annuli" that have inner radius one in $\overline{\mathbb{M}}_{\bigcirc}$.

First, the preimage of U in $\overline{\mathbb{M}}_{\bigcirc}$ may contain configurations with annuli that don't correspond to vertices of T_i. There is a retraction onto the subspace containing no such inappropriate annuli. None of these illegal configurations are in the image of $\overline{\mathfrak{m}}_{T_i}$ so we can begin by replacing the preimage with its retract. Let $\mathcal{E}_1, \mathcal{E}_2$ be sets of edges of T_i so that $\mathcal{E}_1 \cup \mathcal{E}_2 = E(T_i)$. Let $V_{\mathcal{E}_1,\mathcal{E}_2} \subset \overline{\mathbb{M}}_{\bigcirc}$ be the the subset of the preimage of U so that the inner radius of every annulus corresponding to an edge in $\mathcal{E}_1$ is greater than zero and the inner radius of every annulus corresponding to an edge in $\mathcal{E}_2$ is less than one. These are open sets and cover the preimage of U, as every radius is either greater than zero or less than one. They are closed under intersection, as $V_{\mathcal{E}_1,\mathcal{E}_2} \cap V_{\mathcal{E}_1',\mathcal{E}_2'} = V_{\mathcal{E}_1\cup\mathcal{E}_1',\mathcal{E}_2\cup\mathcal{E}_2'}$. So by Lemma 7.4, it suffices to show that the map from the preimage in $\overline{\mathfrak{m}}_{T_i}$ of $V_{\mathcal{E}_1,\mathcal{E}_2}$ is a weak equivalence.

In both the domain and the range there is a deformation retraction to the subset where the radius of every annulus corresponding to an edge in $\mathcal{E}_1 \backslash \mathcal{E}_2$ has radius one. This involves making a choice; we shall hold the outer radius fixed and increase the inner radius, modifying the little disk inside it. Now it suffices to show that the map in question is a weak equivalence when restricted to these deformation retracts.

In this restricted situation, we have specified center and inner and outer radii for each of the edges in $\mathcal{E}_2$, so to specify a section, it suffices to choose a center and outer radius for each edge in $\mathcal{E}_1 \backslash \mathcal{E}_2$ (the inner radius must be one).

We will use the centers of mass L_e as centers. We can construct r_{inf} and r_{sup} as in the construction of Γ for the edges in $\mathcal{E}_1 \backslash \mathcal{E}_2$, using the fixed inner radii of the other annuli in place of r_{inf} and r_{har} (respectively the outer radii for r_{ari}). We choose only one of the radii, say, r_{har}, for the edges corresponding to annuli of inner radius one, using it as both r_{har} and r_{ari}.

The homotopy used in the proof of Lemma 7.14 can be used without modification because it preserves annuli of inner radius one, annuli with inner radius less than one, and annuli with inner radius greater than zero. $\square$

Proof of Theorem 7.3. Let U be a finite intersection of some collection of T-neighborhoods in $\overline{\mathcal{M}}$. By Lemmas 7.6 and 7.14, the maps from the preimages of U in $\overline{\mathfrak{m}}_{T_i}$ to U are weak equivalences. By Lemma 7.19 and the two-out-of three axiom, the maps from the preimages of U in $\overline{\mathbb{M}}_{\bigcirc}$ to U are weak equivalences.

Neighborhoods of the form U form a cover of $\overline{\mathcal{M}}$ with the finite intersection property. Thus by Lemma 7.4, $\overline{\mathbb{M}}_{\bigcirc} \to \overline{\mathcal{M}}$ is a weak equivalence. $\square$

Appendix A. Model category background

We briefly describe model categories, which are used to define the homotopy pushout. [Hov99] and [Hir02] are good general references; we will not state all of the background to the theory but instead will use several results as a black box.

Definition A.1. Let $\mathfrak{f}$ and $\mathfrak{g}$ be morphisms in a category. We say that $\mathfrak{f}$ has the left lifting property with respect to $\mathfrak{g}$ and that $\mathfrak{g}$ has the right lifting property with respect to $\mathfrak{f}$ if there is a dotted filler morphism for every solid diagram of the form

$$\begin{tikzcd} \mathcal{W} \arrow[r] \arrow[d, "\mathfrak{f}"'] & \mathcal{X} \arrow[d, "\mathfrak{g}"] \\ \mathcal{Y} \arrow[r] \arrow[ur, dotted] & \mathcal{Z} \end{tikzcd}$$

Definition A.2. A *model category* is a bicomplete category equipped with three subcategories, called *fibrations*, *cofibrations*, and *weak equivalences* satisfying the axioms below. A morphism that is both a weak equivalence and in one of the other two subcategories is called *trivial.* An object in the model category is called *cofibrant* if the morphism from the initial object to it is a cofibration and *fibrant* if the morphism to the terminal object is a fibration. The axioms are:

(1) all classes are closed under retracts,
(2) (two of three) if $\mathfrak{f}$ and $\mathfrak{g}$ are composable morphisms so that two of $\mathfrak{f}$, $\mathfrak{g}$, and $\mathfrak{f} \circ \mathfrak{g}$ are weak equivalences, so is the third.
(3) A morphism is a (trivial) cofibration if and only if it has the left lifting property with respect to all trivial fibrations (fibrations), and a morphism is a (trivial) fibration if and only if it has the right lifting property with respoct to all trivial cofibrations (cofibrations). Thus the weak equivalences and the fibrations determine the cofibrations, among other similar statements.
(4) Every morphism can be factored as a cofibration followed by a fibration, either one of which may be chosen to be a weak equivalence.

We will need the following facts:

Fact 2. (1) *If*

$$\begin{tikzcd} \mathcal{W} \arrow[r] \arrow[d, "\mathfrak{f}"'] & \mathcal{X} \arrow[d, "\mathfrak{g}"] \\ \mathcal{Y} \arrow[r] & \mathcal{Z} \end{tikzcd}$$

is a pushout diagram and $\mathfrak{f}$ *is a cofibration, so is* $\mathfrak{g}$

(2) *The category of (weak Hausdorff) compactly generated spaces is a model category where the fibrations are the Serre fibrations and the weak equivalences are maps inducing isomorphisms on all homotopy groups.*

(3) *Let $W \to X$ and $Y \to Z$ be cofibrations of spaces. Then $W \times Z \sqcup_{W \times Y} X \times Y \to X \times Z$ is a cofibration.*

This last has two useful corollaries:

Corollary A.3. (1) *Let $W \to X$ be a cofibration of spaces and Z a cofibrant space. Then $W \times Z \to X \times Z$ is a cofibration of spaces.*
(2) *Let X and Z be cofibrant spaces. Then $X \times Z$ is cofibrant.*

We will use several special Reedy categories in passing. These are diagram categories which are well-behaved with respect to model categories. More details about this tool can be found in Chapter 15 of [Hir02]

Definition A.4. Let $\mathbb{N}$ denote the category whose objects are natural numbers and where there is a unique morphism $m \to n$ if $m \leq n$. A (small) *direct category* is a small category equipped with a functor to $\mathbb{N}$ that takes nonidentity morphisms to nonidentity morphisms.

Definition A.5. A *nearly direct category* is a small category $\mathcal{C}$ equipped with a direct subcategory so that the complement of the direct subcategory contains at most one morphism and is closed under composition. For example, any direct category is nearly direct.

Proposition A.6. *[*[Hir02]*, Theorems 15.3.4 and 15.10.9] Let $\mathcal{C}$ be a model category and $\mathcal{D}$ a nearly direct category. Then the category $\mathcal{C}^{\mathcal{D}}$ of $\mathcal{D}$-diagrams in $\mathcal{C}$ has the structure of a model category where the weak equivalences are objectwise weak equivalences of diagrams and cofibrant objects are diagrams with every object cofibrant in $\mathcal{C}$ and every morphism in the direct subcategory a cofibration. If $\mathcal{D}$ is a direct category, then the fibrations are objectwise fibrations.*

In either case, the colimit functor $\mathcal{C}^{\mathcal{D}} \to \mathcal{C}$ takes cofibrations and weak equivalences between cofibrant objects to cofibrations and weak equivalences in $\mathcal{C}$.

Lemma A.7. *The following are direct categories:*
(1) *The deleted S-cube category $\mathbf{Cu}_S$ (see Definition 6.1)*
(2) *The pushout category $\bullet \leftarrow \bullet \to \bullet$ (this is the deleted $\{0,1\}$-cube category).*
(3) *The telescope category $\bullet \to \bullet \to \bullet \to \cdots$*

The pushout category is also a nearly direct category, with direct subcategory missing one of the two non-identity morphisms.

Definition A.8. Let $\mathcal{D}$ be the pushout category, considered as a nearly direct category. Let $\mathcal{W}$ be a pushout diagram in a model category $\mathcal{C}$.
(1) A *model of* $\mathcal{W}$ is a diagram $\overline{\mathcal{W}}$ equipped with a morphism $\overline{\mathcal{W}} \to \mathcal{W}$ which is a weak equivalence in $\mathcal{C}^{\mathcal{D}}$ (an objectwise weak equivalence). Its *realization* is its colimit.
(2) A *cofibrant model (realization) of the homotopy pushout* of $\mathcal{W}$ is a model of $\mathcal{W}$ that is cofibrant in $\mathcal{C}^{\mathcal{D}}$ (its realization)
(3) A *model (realization) of the homotopy pushout* of $\mathcal{W}$ is a model of $\mathcal{W}$ which accepts a weak equivalence from a cofibrant model that passes to a weak equivalence of realizations.
(4) A *weak realization of the homotopy pushout* of $\mathcal{W}$ is an object equipped with a weak equivalence to the realization of a model of the homotopy pushout of $\mathcal{W}$

Lemma A.9. *Let $\mathcal{W} \to \mathcal{W}'$ be a weak equivalence of pushout diagrams in a model category $\mathcal{C}$.*

(1) *There is a zigzag of weak equivalences over $\mathcal{W}$ of any two cofibrant models of the homotopy pushout of $\mathcal{W}$ inducing weak equivalences of their realizations.*
(2) *There is a zigzag of weak equivalences over $\mathcal{W}'$ between any pair of cofibrant models of the homotopy pushouts of $\mathcal{W}$ and $\mathcal{W}'$*
(3) *There is a zigzag of weak equivalences between any pair of weak realizations of the homotopy pushouts of $\mathcal{W}$ and $\mathcal{W}'$.*

Proof. (1) Let $\widetilde{\mathcal{W}}$ and $\widetilde{\mathcal{W}}'$ be two cofibrant models for $\mathcal{W}$. Factorize $\widetilde{\mathcal{W}} \to \mathcal{W}$ as a trivial cofibration followed by a trivial fibration. The intermediate diagram is cofibrant and there is a lift from $\widetilde{\mathcal{W}}'$ into it over $\mathcal{W}$. Everything is a weak equivalence by the two-of-three axiom and the weak equivalences pass to realizations by Proposition A.6.
(2) Let $\widetilde{\mathcal{W}}$ and $\widetilde{\mathcal{W}}'$ be the two cofibrant models. Then $\widetilde{\mathcal{W}}$ is also a cofibrant model for $\mathcal{W}'$. Factorize as before.
(3) This follows from the first parts of the lemma and the definition.

□

Appendix B. Trees

We fix the conventions used for trees.

Definition B.1. Let S be a finite set and let $\mathscr{P}_\star(S)$ denote the *nonempty* subsets of S. An S-tree T consists of a finite subset E of $\mathscr{P}_\star(S) \times \mathbb{N}$, called *edges*, so that:

(1) For each s in S, $(\{s\}, 0)$ is in E,
(2) $(S, 0)$ is in E,
(3) if (P, m) is in E for $m > 0$, then $(P, m-1)$ is in E, and
(4) if $(P, 0)$ and $(P', 0)$ are in E then either P and P' have empty intersection or one contains the other.

The set of edges of the form $(\{s\}, 0)$ are called leaves; the edge of form (S, m), for maximal m, is called the *root*. Collectively the edges and root are called *external edges*. The leaf and root may coincide when $|S| = 1$. All other edges are called *internal edges*.

We call the edges which are not leaves *vertices*. E forms a poset by inclusion of subsets of S and $<$ on $\mathbb{N}$. The conditions imply that every edge except the root has a unique successor. For e an edge, we call e, viewed as a vertex or leaf, the *input* and its successor (if it exists) the *output* of the edge; Note that if $|S| > 1$ the number of vertices is one greater than the number of internal edges. A tree with only one vertex is called a *corolla*.

A tree is *stable* if every edge is of the form $(P, 0)$, in which case we will omit 0 in notation for edges, vertices, and leaves. A tree is *nearly stable* if every edge is of the form $(P, 0)$, $(\{s\}, 1)$, or $(S, 1)$.

Call the set of stable S-trees $\mathbb{T}(S)$ and stable S-trees with n vertices $\mathbb{T}_n(S)$. For $T \in \mathbb{T}_n(S)$, let $V(T)$, $E(T)$, $E^{\text{ext}}(T)$, and $E^{\text{int}}(T)$ denote the vertices, edges, external edges, and internal edges of T, and let $E(v)$ (similarly $E^{\text{ext}}(v), E^{\text{int}}(v)$) denote the edges which have v as output.

Definition B.2. Let T be a tree and (P, m) an edge. The tree obtained from T by *inserting a vertex* on (P, m) has the same edges as T, except that there is a new

edge of the form (P, n) where n is the lowest number that makes this a new edge. An edge of the form (P, i) for $i > m$ in T is identified with the edge $(P, i+1)$ in the insertion tree, and all other edges are identified with themselves.

Definition B.3. Let T be a tree and $e = (P, m)$ an internal edge of T. The tree T_e obtained by *contracting* the edge e is as follows. Let n be the highest number so that there is an edge of form (P, n). Then T_e has edges $E(T)\backslash(P, n)$. Any edge of the form (P, i) for $i > m$ in T is identified with the edge $(P, i-1)$ in T_e, and all other edges are identified with themselves, except (P, m), the *contraction edge*.

The vertex which is the image of the output of e is called the *contraction vertex*. We consider it to be the identification of the output and input of e. If these are v and v', we write this contraction vertex $\{v, v'\}$.

If e is of the form (P, m) for $m > 0$, then contracting e is also called *forgetting the vertex* e.

If $\mathcal{E}$ is a set of internal edges, then $T_{\mathcal{E}}$ is the tree obtained by contracting all the edges in $\mathcal{E}$. The order of contraction does not matter.

The set $\mathbb{T}(S)$ form the objects of a (small) category where the morphisms out of a tree T are indexed by sets of internal edges: $\mathcal{E}$ is a morphism from T to $T_{\mathcal{E}}$. Composition uses the identification above to define a bigger contraction set.

Definition B.4. Let T and T' be S and S'-trees. Let λ be a leaf of T, and let $f : S' \sqcup S\backslash\{\lambda\} \to S''$ be a relabelling isomorphism. Then the S''-tree obtained by *grafting* T' to T at the leaf λ has edges

(1) $(f(P), i)$ for $(P, i) \in E(T')$,
(2) $(f(P), i)$ for $(P, i) \in E(T)$ if $\lambda \notin P$,
(3) $(f(P\backslash\{\lambda\}) \cup f(S'), i)$ for $(P, i) \in E(T)$ if $\{\lambda\} \subsetneq P$, and
(4) $(f(S'), i+n)$ for $(\{\lambda\}, i)$ in $E(T)$, where (S', n) is the root of T'.

The root of T' and leaf $(\{\lambda\}, 0)$ of T both yield the same edge of T'' under this identification; this edge is called the *grafting edge*.

Appendix C. Operads

This appendix contains a very brief background on operads. In this section, $\mathcal{C}$ will denote either the category of compactly generated spaces or the category of sets. The constructions would be valid in many other categories. This material is all standard; references include [May72, Smi85, GJ94]

Definition C.1. A *collection* in $\mathcal{C}$ is a functor from the category whose objects are finite sets and whose morphisms are isomorphisms of finite sets to $\mathcal{C}$.

The *unit collection* I has $I(S)$ equal to S (with the discrete topology if appropriate) if $|S| = 1$ and $I(S)$ the empty set or space. A *pointed collection* is a collection $\mathcal{U}$ equipped with a natural transformation $I \to \mathcal{U}$; that is, the category of pointed collections is the undercategory of I.

Definition C.2. For a fixed finite set S, Let $\mathcal{D}_S$ be the groupoid of the undercategory of S in the category whose objects are finite sets and whose morphisms are maps of finite sets. An object of $\mathcal{D}_S$ consists of a finite set S' and a map $f : S \to S'$. A morphism is a commuting isomorphism of finite sets.

Let $\mathcal{U}$ and $\mathcal{V}$ be collections in category. Then the product $\mathcal{U} \boxtimes \mathcal{V}$ is a collection defined as

$$(\mathcal{U} \boxtimes \mathcal{V})(S) = \operatorname*{colim}_{\mathcal{D}_S} \left(\mathcal{U}(S') \times \prod_{s \in S'} \mathcal{V}(\{f^{-1}(s)\}) \right).$$

Lemma C.3. *$\boxtimes$ and I give the category of collections the structure of a monoidal category.*

Definition C.4. An *operad* is a monoid $\mathcal{O}$ in the monoidal category of collections. That is, there is a product $\circ$ from $\mathcal{O} \boxtimes \mathcal{O} \to \mathcal{O}$ and a unit $I \to \mathcal{O}$ satisfying associativity and unit constraints.

We will assume throughout the paper that all operads are constructed with $\mathcal{C}$ the category of compactly generated spaces unless otherwise stated.

Definition C.5. If $\mathcal{O}$ is an operad, let $\mathcal{O}(n)$ denote $\mathcal{O}(\{1, \ldots, n\})$. We will define *partial composition maps* $\circ_i : \mathcal{O}(m) \times \mathcal{O}(n) \to \mathcal{O}(m+n-1)$, where $1 \leq i \leq m$. These are defined in terms of the monoidal product of the operad $\mathcal{O}$ as follows.

$\mathcal{O}(m) \times \mathcal{O}(n)$ is canonically isomorphic to

$$\mathcal{O}(m) \times \mathcal{O}(1) \times \cdots \times \mathcal{O}(1) \times \mathcal{O}(n) \times \mathcal{O}(1) \times \cdots \times \mathcal{O}(1).$$

Fix the map $f : \{1, \ldots, m+n-1\} \to \{1, \ldots, m\}$ given by

$$f(j) = \begin{cases} j, & j < i \\ i, & i \leq j < i+n \\ j-n+1, & i+n \leq j < m+n \end{cases}$$

Then there is a factor in the colimit defining $(\mathcal{O} \boxtimes \mathcal{O})(m+n-1)$ that looks like

$$\mathcal{O}(m) \times \mathcal{O}(\{1\}) \times \mathcal{O}(\{2\}) \times \cdots \times \mathcal{O}(\{i, \ldots, i+n-1\}) \times \cdots \times \mathcal{O}(\{m+n-1\}).$$

There is an evident map from $\mathcal{O}(m) \times \mathcal{O}(n)$ into this and a structure map from it to $\mathcal{O}(m+n-1)$, as desired.

The following is well-known:

Lemma C.6. *To give the structure of an operad, it suffices to define the unit $I \to \mathcal{O}(1)$ and the structure maps $\circ_i$*

There is nothing special about the sets $\{1, \ldots, n\}$:

Definition C.7. Let S, S', and S'' be finite sets, let $s \in S$, and let $f : S \backslash \{s\} \sqcup S' \to S''$ be an isomorphism. Then there is a *partial composition map* $\circ_{f,s}$ constructed in a similar manner, where we take the factor in the colimit corresponding to the map taking $f(S') \subset S''$ to $s \in S$.

Definition C.8. For a S-tree T and an operad $\mathcal{O}$, let $\mathcal{O}(T)$ be the product over the vertices $\prod_{v \in V(T)} \mathcal{O}(E(v))$. For e an internal edge of T, define a map $\mathcal{O}(T) \to \mathcal{O}(T_e)$ as follows. Most vertices of T_e correspond to a single vertex of T, with a canonical isomorphism of the incoming edge set which induces an isomorphism of the corresponding operadic space. If v and v' are the vertices involved in the contraction, and e is the outgoing edge from v' to v, then contraction defines an isomorphism $f : E(\{v, v'\}) \to E(v) \backslash \{e\} \sqcup E(v')$ which gives a partial composition map as in the previous definition. The *composition along a tree* $\mathcal{O}(T) \to \mathcal{O}(S)$ is given by repeated contraction followed by the canonical isomorphism induced by

that between the incoming edges of the S-corolla and S. The conditions defining an operad imply that this is well-defined.

Appendix D. Proof of Proposition 7.18

Proposition 7.18 says that the homotopy H lands in $\overline{\mathfrak{m}}_T^{\mathrm{Top}}$. The content of this proposition is that the homotopy preserves the configuration in which disks are nested in one another. That is, if X_0 and X_1 are properly overlapping disks (and likewise Y_0 and Y_1, then:

(1) If X_i is contained in Y_i for $i \in \{0,1\}$, then for all t, the disk which is the image of the pair X_i under the homotopy slice H_t is contained in the disk which is the image of the pair Y_i under the homotopy slice H_t.
(2) If X_i and Y_i are disjoint for $i \in \{0,1\}$, then for all t, the disks which are the images under the homotopy slice H_t of the pairs X_i and Y_i are disjoint.

These two statements are, respectively, Corollaries D.8 and D.12, and by proving them, we prove the proposition.

Lemma D.1. *$h_t(c_0, r_0, c_1, r_1, i)$ coincides with (c_i, r_i) for $i = 0, 1$, justifying the notation (c_t, r_t). The disk centered at c_t of radius r_t contains the intersection of the disks centered at c_i of radius r_i, for $i \in 0, 1$.*

Proof. The first statement follows immediately from the definition. Let p be a point in the intersection. We will prove the second statement in the case that c_0, c_1, and p are disjoint; the other cases are easier. In this case, there is a (possibly degenerate) triangle with corners c_0, c_1, and p. c_t is on the segment between c_0 and c_1. Repeated applications of the law of cosines and substitution show that the distance from c_t to p satisfies

$$|c_t - p|^2 = \frac{|c_0 - p|^2|c_1 - c_t| + |c_1 - p|^2|c_0 - c_t|}{|c_0 - c_1|} - |c_0 - c_t||c_1 - c_t|.$$

Using the facts that $|c_0 - p| \leq r_0$ and $|c_1 - p| \leq r_1$, the definitions of θ and c_t, and trigometric identities, we can use this to generate the inequality

$$|c_t - p|^2 \leq r_t^2 \left(2\cos\theta \sin(t\theta)\sin((1-t)\theta) + \sin^2(t\theta) + \sin^2((1-t)\theta)\right) = r_t^2 \sin^2\theta,$$

which is less than or equal to r_t^2. □

Lemma D.2. *Let $|c_0 - c_1| < r_0 + r_1$. Then:*

(1) $c_{\frac{1}{2}} = \frac{c_1 r_0 + c_0 r_1}{r_0 + r_1}$,
(2) $r_{\frac{1}{2}} = \frac{2r_0 r_1 \cos(\frac{\theta}{2})}{r_0+r_1} = \frac{\sqrt{r_0 r_1}\sqrt{(r_0+r_1)^2 - |c_0-c_1|^2}}{r_0+r_1}$,
(3) $\theta(c_0, c_{\frac{1}{2}}, r_0, r_{\frac{1}{2}}) = \frac{\theta}{2} = \theta(c_{\frac{1}{2}}, c_1, r_{\frac{1}{2}}, r_1)$, *and*
(4) $h(c_0, c_{\frac{1}{2}}, r_0, r_{\frac{1}{2}}, 2t) = h(c_0, c_1, r_0, r_1, t) = h(c_{\frac{1}{2}}, c_1, r_{\frac{1}{2}}, r_1, 2t-1)$.

These are all straightforward evaluations, with some trigonometric substitutions

Definition D.3. Let X and Y be disks. The *nesting distance* $d(X,Y)$ is $r_Y - r_X - |c_X - c_Y|$. This quantity is nonnegative (positive) if X is (properly) nested in Y; in this case it is the distance from the interior of X to the exterior of Y.

Remark 2. By the triangle inequality the nesting distance is superadditive; that is, $d(X,Z) \geq d(X,Y) + d(Y,Z)$.

Notation D.4. If $X_0 = (c_0, r_0)$ and $X_1 = (c_1, r_1)$ are properly overlapping disks, then we will use X_t to refer to the disk (c_t, r_t) determined from these two by h.

Proposition D.5. *Let $\{X_0, X_1\}$ and $\{Y_0, Y_1\}$ be two pairs of properly overlapping disks. Suppose X_i is nested in Y_i for $i \in \{0,1\}$. Then $X_{\frac{1}{2}}$ is nested in $Y_{\frac{1}{2}}$, and, in fact, $d(X_{\frac{1}{2}}, Y_{\frac{1}{2}})$ is greater than or equal to the minimum of $d(X_0, Y_0)$ and $d(X_1, Y_1)$. Furthermore, $d(X_{\frac{1}{2}}, Y_{\frac{1}{2}})$ can only be zero if* both *$d(X_0, Y_0)$ and $d(X_1, Y_1)$ are zero.*

Lemma D.6. *Proposition D.5 is true in the special case where X_0 and Y_0 have the same center, c_0, and X_1 and Y_1 have the same center, c_1.*

Proof. In this case we can write

$$c(X_0, X_1, \frac{1}{2}) - c(Y_0, Y_1, \frac{1}{2}) = \frac{(c_1 - c_0)(r_{X_0} r_{Y_1} - r_{X_1} r_{Y_0})}{(r_{X_0} + r_{X_1})(r_{Y_0} + r_{Y_1})}$$

Reparameterize using the variables

$$\begin{aligned} u &= |c_0 - c_1| \\ w &= r_{X_0} + r_{X_1} \\ x &= r_{X_0} - r_{X_1} \\ y &= r_{Y_0} + r_{Y_1} \\ z &= r_{Y_0} - r_{Y_1}. \end{aligned}$$

Note that w and y are always positive, $y \geq w > u$, that $-w < x < w$, and that $-y < z < y$. In fact, $w + x - y \leq z \leq -w + x + y$. Then we can define $g(u, w, x, y, z) = d(X_{\frac{1}{2}}, Y_{\frac{1}{2}}) - \min\{d(X_0, Y_0), d(X_1, Y_1)\}$ and write

$$g = \frac{\sqrt{y^2 - z^2}\sqrt{y^2 - u^2}}{2y} - \frac{\sqrt{w^2 - x^2}\sqrt{w^2 - u^2}}{2w} - \frac{u|xy - wz|}{2wy} + \frac{w - y}{2} + \frac{|x - z|}{2}.$$

This is piecewise differentiable with respect to z with derivative

$$g_z = -\frac{z\sqrt{y^2 - u^2}}{2y\sqrt{y^2 - z^2}} \pm \frac{u}{2y} \pm \frac{1}{2}$$

This is really four different functions, depending on the signs of $\frac{u}{2y}$ and $\frac{1}{2}$. Each of the four functions is defined globally. For each of them, as z approaches $-y$ g_z is positive and as z approaches y, g_z is negative. Further, each of these four globally defined functions has a single zero (at $z = \pm\sqrt{\frac{y(y \pm u)}{2}}$), which may or may not be in the appropriate domain for that choice of signs. This means that any zero of the overall piecewise function g_z is a local maximum, so local minima can only occur at boundary points and at the discontinuities of the derivative, which occur at $z = x$ and $z = \frac{y}{w}x$.

At the discontinuity $z = \frac{y}{w}x$, say that the function changes from $g_{z,1}$ to $g_{z,2}$. If $x \neq 0$, then the zero of $g_{z,1}$ is at least as great as the zero of $g_{z,2}$. So either the global function g_z increases to the left of $z = \frac{y}{w}x$, decreases to the right of it, or both, and in particular, this cannot be a minimum of g. If $x = 0$ then $\frac{y}{w}x$ coincides with x.

To summarize the progress at this stage of the proof, we have shown that holding the first four variables constant, g achieves its minimal value for z in the interval $[w + x - y, -w + x + y]$ at either one of the endpoints or the midpoint $z = x$. We will show that in these minimal cases, $g \geq 0$.

At the endpoints $z = x \pm (w - y)$, the function g is continuously differentiable in y, with positive derivative. We have shown the desired inequality; it remains to show that $d(X_{\frac{1}{2}}, Y_{\frac{1}{2}})$ can only be zero if *both* $d(X_0, Y_0)$ and $d(X_1, Y_1)$ are zero. By the definition of g, this can only happen if g is zero and one of the two quantities $d(X_0, Y_0)$ and $d(X_1, Y_1)$ is zero. This can only occur in two cases. In the first case, $w = y$ and $x = z$. In this case X_i coincides with Y_i for both $i = 0$ and $i = 1$. In the second case, $x = z$ and $u = |x|$. In this case, without loss of generality, if $d(X_0, Y_0)$ is zero then $r_{X_0} = r_{Y_0}$ and then $z = x$ implies $r_{X_1} = r_{Y_1}$ so $d(X_1, Y_1) = 0$ as well. □

Lemma D.7. *Proposition D.5 is true in the special case where X_0 and Y_0 are tangent and X_1 and Y_1 coincide. The proposition is also true in the special case where X_0 and Y_0 coincide and X_1 and Y_1 are tangent.*

Proof. We prove only the first statement; the second statement's proof is essentially identical. If the center c_Y of Y_0 coincides with either the center c_X of X_0 or the center c_Z of X_1 and Y_1, then the proof can be completed by simple algebraic manipulation. If c_Y is distinct from both c_X and c_Z, we can parameterize with:

$$\begin{aligned} u &= |c_Y - c_Z| \\ w &= r_{X_0} + r_{X_1} \\ x &= r_{X_1} = r_{Y_1} \\ y &= r_{Y_0} + r_{Y_1} \\ z &= \cos\theta, \end{aligned}$$

where θ is the angle formed at c_Y by the rays toward c_X and c_Z. Note that $|c_X - c_Y| = y - w$ and then by the law of cosines,

$$|c_X - c_Z| = \sqrt{u^2 + (y-w)^2 - 2u(y-w)z}$$

and $|c_{X_{\frac{1}{2}}} - c_{Y_{\frac{1}{2}}}|$ is $\frac{x(y-w)\sqrt{u^2+y^2-2uyz}}{wy}$. These formulae are still valid if c_Y coincides with c_X or c_Z, even though z is no longer well-defined in these cases.

Now, defining $g(u, w, x, y, z)$ as in the previous lemma with these different variables, we can differentiate with respect to z, and find that g has a unique global minimum at

$$z_0 = \frac{y^2 + u^2 - 2xy}{2u(y-x)},$$

which may or may not be in $[-1, 1]$. Whether or not it is, $g(z) \geq g(z_0) = 0$, with equality only at $z = z_0$.

□

Proof of Proposition D.5. Let $\{X_0, X_1, Y_0, Y_1\}$ be disks as in the hypotheses of the proposition. Then X_0 is contained in a disk X_0' with the same center which is tangent to and contained in Y_0, and likewise X_1 is contained in a disk X_1' with the same center which is tangent to and contained in Y_1. Then by Lemma D.6, $d(X_{\frac{1}{2}}, X'_{\frac{1}{2}}) \geq \min\{d(X_0, X_0'), d(X_1, X_1')\}$. If $Z_0 = Y_0$ and $Z_1 = X_1'$, then Lemma D.7 shows that $X'_{\frac{1}{2}}$ is nested in $Z_{\frac{1}{2}}$ and that $Z_{\frac{1}{2}}$ is nested in $Y_{\frac{1}{2}}$. So

$$\begin{aligned} d(X_{\frac{1}{2}}, Y_{\frac{1}{2}}) &\geq d(X_{\frac{1}{2}}, X'_{\frac{1}{2}}) \\ &\geq \min\{d(X_0, X_0'), d(X_1, X_1')\} \geq \min\{d(X_0, Y_0), d(X_1, Y_1)\} \end{aligned}$$

$d(X_{\frac{1}{2}}, X'_{\frac{1}{2}})$ is greater than zero unless X_i and X'_i coincide for both $= i = 0$ and $i = 1$, in which case X_i and Y_i coincide or are tangent for both $i = 0$ and $i = 1$. □

Corollary D.8. *Let $\{X_0, X_1, Y_0, Y_1\}$ be as in Proposition D.5. Then X_t is nested in Y_t for all $t \in [0,1]$, that is, $d(X_t, Y_t) \geq 0$. This inequality is strict for $t \in (0,1)$ unless $d(X_0, Y_0) = d(X_1, Y_1) = 0$.*

Proof. We begin by showing the inequality for rational t of the form $\frac{k}{2^m}$ (*dyadic rationals*). We proceed by induction on m. If $m = 0$, the statement is true by assumption. Suppose we have shown the statement for $m < n$. By repeated applications of Lemma D.2 and Proposition D.5, the statement is true for $m = n$. Since being nested is a closed condition, being true on a dense subset of the interval implies that it is true throughout the interval.

Now assume, without loss of generality, that $d(X_0, Y_0) > 0$. By the same induction, $d(X_t, Y_t) > 0$ for dyadic rationals except possibly at $t = 1$. Then for irrational t, pick two consecutive such rationals bracketing it: $\alpha = \frac{k}{2^m} < t < \gamma = \frac{k+1}{2^m} < 1$. Then as the induction proceeds, for all dyadic rationals β between α and γ, $d(X_\beta, Y_\beta) \geq \min\{d(X_\alpha, Y_\alpha), d(X_\gamma, Y_\gamma)\} > 0$. Then this inequality holds in the limit and $d(X_t, Y_t)$ is bounded away from zero. □

Definition D.9. Let $\mathbb{H}$ be the space of half-planes in the standard plane, identified with $S^1 \times \mathbb{R}$ in the following manner: the pair (θ, ρ) consists of points in the plane whose inner product with $(\cos\theta, \sin\theta)$ is greater than (or greater than or equal to) ρ. We will not distinguish between open and closed half-planes, which are in one-to-one correspondence. Let the space of *admissible pairs of half-planes*,$\mathbb{A} \subset \mathbb{H}^2$ be the subspace of points $(\theta_1, \rho_1, \theta_2, \rho_2)$ where $\theta_1 \neq -\theta_2$.

We define the *intermediate half plane* of an admissible pair as a map $\mathbb{A} \to \mathbb{H}$ defined as

$$(\theta_1, \rho_1, \theta_2, \rho_2) \mapsto \left(\frac{\theta_1 + \theta_2}{2}, \frac{\rho_1 + \rho_2}{2\cos\frac{\theta_1 - \theta_2}{2}} \right)$$

where representatives of θ_1 and θ_2 are chosen so that they differ by less than π (this ensures that the halves are well-defined).

In words, if the bounding lines of the two half-planes intersect, then the bounding line of the intermediate half-plane goes through their intersection, bisecting one of the pairs of vertical angles. If the bounding lines are parallel, then the intermediate half-plane is parallel to both and half-way between them. In either case, the intermediate half-plane contains the intersection of the two half-planes of the admissible pair.

Proposition D.10. *Let X_0 and X_1 be properly overlapping disks. Choose a half plane $\mathcal{H}_i$ containing X_i whose boundary line is tangent to X_i. If the pair $(\mathcal{H}_0, \mathcal{H}_1)$ is admissible, then $X_{\frac{1}{2}}$ is contained in the intermediate half-plane of the pair.*

Proof. Let $\mathcal{H}_i = (\theta_i, \rho_i)$. It suffices to show that

$$\left\langle \frac{c_1 r_0 + c_0 r_1}{r_0 + r_1}, \left(\cos\left(\frac{\theta_0 + \theta_1}{2}\right), \sin\left(\frac{\theta_0 + \theta_1}{2}\right) \right) \right\rangle \\ \geq \frac{\rho_0 + \rho_1}{2\cos\left(\frac{\theta_0 - \theta_1}{2}\right)} + \sqrt{r_0 r_1} \frac{\sqrt{(r_0 + r_1)^2 - |c_0 - c_1|^2}}{r_0 + r_1}$$

By assumption, $\langle c_i, (\cos\theta_i, \sin\theta_i)\rangle \geq \rho_i + r_i$. Using this and the identity

$$\cos\left(\frac{\theta_0+\theta_1}{2}\right) = \frac{\cos\theta_0 + \cos\theta_1}{2\cos\left(\frac{\theta_0-\theta_1}{2}\right)},$$

along with the cognate identity for sine, means that it suffices to show

$$\begin{aligned}&\langle c_1 r_0 + c_0 r_1, (\cos\theta_0 + \cos\theta_1, \sin\theta_0 + \sin\theta_1)\rangle \\ &\qquad \geq (r_0+r_1)\left(-r_0 - r_1 + \langle c_0, (\cos\theta_0, \sin\theta_0)\rangle + \langle c_1, (\cos\theta_1, \sin\theta_1)\rangle\right) \\ &\qquad\qquad + 2\sqrt{r_0 r_1}\sqrt{(r_0+r_1)^2 - |c_0 - c_1|^2}\cos\left(\frac{\theta_0-\theta_1}{2}\right).\end{aligned}$$

Rearranging, this is the same as

$$\begin{aligned}&(r_0+r_1)^2 + r_1\langle c_0 - c_1, (\cos\theta_1, \sin\theta_1)\rangle + r_0\langle c_1 - c_0, (\cos\theta_0, \sin\theta_0)\rangle \\ &\qquad \geq \sqrt{4r_0 r_1}\sqrt{(r_0+r_1)^2 - |c_0 - c_1|^2}\cos\left(\frac{\theta_0-\theta_1}{2}\right).\end{aligned}$$

If $c_0 = c_1$, then squaring both sides clearly yields the desired inequality. So assume that $c_0 \neq c_1$, so that we can write $c_0 - c_1 = |c_0 - c_1|(\cos\nu, \sin\nu)$ for some angle ν. Then only the second and third terms have any dependence on ν; explicitly they are:

$$|c_0 - c_1|(r_1\cos\nu\cos\theta_1 + r_1\sin\nu\sin\theta_1 - r_0\cos\nu\cos\theta_0 - r_0\sin\nu\sin\theta_0)$$

this quantity reaches its maximum and minimum values when

$$\cos\nu = \pm\frac{r_0\cos\theta_0 - r_1\cos\theta_1}{\sqrt{r_0^2 + r_1^2 - 2r_0 r_1\cos(\theta_0-\theta_1)}};\ \sin\nu = \pm\frac{r_0\sin\theta_0 - r_1\sin\theta_1}{\sqrt{r_0^2 + r_1^2 - 2r_0 r_1\cos(\theta_0-\theta_1)}},$$

with the same sign in both cases. The minimum value is

$$-\sqrt{r_0^2 + r_1^2 - 2r_0 r_1\cos(\theta_0-\theta_1)},$$

so it suffices to show the inequality

$$\begin{aligned}&(r_0+r_1)^2 - \sqrt{r_0^2 + r_1^2 - 2r_0 r_1\cos(\theta_0-\theta_1)}|c_0 - c_1| \\ &\qquad \geq \sqrt{4r_0 r_1}\sqrt{(r_0+r_1)^2 - |c_0 - c_1|^2}\cos\left(\frac{\theta_0-\theta_1}{2}\right).\end{aligned}$$

The left hand side of this inequality is nonnegative, since each of the factors in the second term is less than or equal to $r_0 + r_1$, so it suffices to show the square of this inequality. Using $\cos^2\left(\frac{\theta_0-\theta_1}{2}\right) = \frac{1+\cos(\theta_0-\theta_1)}{2}$, and rearranging, this is:

$$(r_0+r_1)^2\left(\sqrt{r_0^2 + r_1^2 - 2r_0 r_1\cos(\theta_0-\theta_1)} - |c_0 - c_1|\right)^2 \geq 0,$$

which is certainly true. □

Corollary D.11. *Let X_0 and X_1 be properly overlapping disks, and let Y_0 and Y_1 be properly overlapping disks. Suppose also that the distances between X_i and Y_i are at least $\epsilon_i > 0$, so that X_i and Y_i do not overlap. Then the distance from $X_{\frac{1}{2}}$ to $Y_{\frac{1}{2}}$ is at least $\frac{\epsilon_0+\epsilon_1}{2}$.*

Proof. Choose the half-plane $\mathcal{H}_{X_i} = (\theta_{X_i}, \rho_{X_i})$ containg X_i whose boundary line is tangent to X_i at the closest boundary point to the center of Y_i, and vice versa. Then $\theta_{X_i} = -\theta_{Y_i}$ and $\mathcal{H}_{X_i}$ and $\mathcal{H}_{Y_i}$ have empty intersection. It is easy to see that if $\theta_{X_0} = -\theta_{X_1}$, that either $\mathcal{H}_{X_0} \cap \mathcal{H}_{X_1}$ or $\mathcal{H}_{Y_0} \cap \mathcal{H}_{Y_1}$ is empty. But $\mathcal{H}_{X_0} \cap \mathcal{H}_{X_1}$ contains $X_0 \cap X_1$, which is nonempty by assumption, and likewise for Y_i. So the pair $\mathcal{H}_{X_i}$ (likewise $\mathcal{H}_{Y_i}$) is admissible.

Then the angles of the intermediate halfplanes $\mathcal{H}_X = (\theta_X, \rho_X)$ and $\mathcal{H}_Y = (\theta_Y, \rho_Y)$ are additive inverses of one another. The distance between these two half-planes is then $\min\{0, \rho_X + \rho_Y\}$. But $\cos \frac{\theta_{X_0} - \theta_{X_1}}{2} = \cos \frac{\theta_{Y_0} - \theta_{Y_1}}{2} > 0$, which implies:

$$\rho_X + \rho_Y = \frac{\rho_{X_0} + \rho_{X_1} + \rho_{Y_0} + \rho_{Y_1}}{2 \cos \frac{\theta_{X_0} - \theta_{X_1}}{2}} \geq \frac{\rho_{X_0} + \rho_{Y_0}}{2} + \frac{\rho_{X_1} + \rho_{Y_1}}{2} = \frac{\epsilon_0}{2} + \frac{\epsilon_1}{2}$$

□

Corollary D.12. *Let X_i, Y_i be as in Corollary D.11. Then X_t and Y_t do not properly overlap one another.*

Proof. Following the same logic as Corollary D.8, by induction if t is a dyadic rational then the distance between X_t and Y_t is at least as great as the lesser distance between X_0 and Y_0 or X_1 and Y_1. Then the same is true for all t since the dyadic rationals are dense. □

References

[BM02] Clemens Berger and Ieke Moerdijk, *Axiomatic homotopy theory for operads*, Comment. Math. Helv. **78** (2002), 805–831.

[BM06] ———, *The Boardman-Vogt resolution of operads in monoidal model categories*, Topology **45** (2006), 807–849.

[BV73] J.M Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Math., vol. 347, Springer, 1973.

[DCV] Gabriel C. Drummond-Cole and Bruno Vallette, *The minimal model for the Batalin-Vilkovisky operad*, arXiv:1105.2008v1.

[DK] Vladimir Dotsenko and Anton Khoroshkin, *Free resolutions via Gröbner bases*, arXiv:0912.4895.

[Get94a] Ezra Getzler, *Batalin-Vilkovisky algebras and two-dimensional topological field theories*, Comm. Math. Phys. **159** (1994), 265–285.

[Get94b] ———, *Two-dimensional topological gravity and equivariant cohomology*, Comm. Math. Phys **163** (1994), 473–489.

[Get95] ———, *Operads and moduli spaces of genus 0 Riemann surfaces*, The moduli space of curves, Progr. Math., vol. 129, Birkhäuser Boston, Boston, Massachusetts, 1995, pp. 199–230.

[GJ94] Ezra Getzler and John D. S. Jones, *Operads, homotopy algebra and iterated integrals for double loop spaces*, arXiv:hep-th/9403055v1, 1994.

[Hir02] Philip S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, Amer. Math. Soc., Providence, Rhode Island, 2002.

[Hov99] Mark Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, Amer. Math. Soc., 1999.

[HV10] V. Hinich. and A. Vaintrob, *Augmented Teichmuller spaces and orbifolds*, Selecta Math., New Series **16** (2010), 533–629.

[Kel98] Bernhard Keller, *On the cyclic homology of ringed spaces and schemes*, Doc. Math. **3** (1998), 231–259.

[Knu83] Finn Knudsen, *The projectivity of the moduli space of stable curves, II: the stacks $M_{g,n}$*, Math. Scand. **52** (1983), 161–199.

[Kon05] Maxim Kontsevich, *On pre-Frobenius manifolds and TQFT in positive genus*, Lecture given at Max Planck Institute, Bonn, Germany, January 2005.

[Mar] Nikita Markarian, *http://nikitamarkarian.wordpress.com/2009/11/22/hycommbv/*.

[May72] J. Peter May, *The geometry of iterated loop spaces*, Lecture Notes in Math., vol. 271, Springer-Verlag, Berlin, Germany, 1972.

[May90] ———, *Weak equivalence and quasifibration*, Groups of Self-Equivalences and Related Topics, Lecture Notes in Math., vol. 1425, Springer, Berlin, 1990, pp. 91–101.

[Smi85] V.A. Smirnov, *Homotopy theory of coalgebras*, Izv. Akad. Nauk SSSR Ser. Mat. 49 **49** (1985), 1302–1321, Translation: Math. USSR Izv. 35 (1990), 445–455.

[Spi01] Markus Spitzweck, *Operads, algebras, and modules in model categories and motives*, Ph.D. thesis, Rheinischen Friedrich-Wilhelms-Universität Bonn, 2001.

[SS03] Stefan Schwede and Brooke Shipley, *Equivalences of monoidal model categories*, Algebraic and Geometric Topology **3** (2003), 287–334.